# An explicit formula for the Hilbert symbol of a formal group

Floric Tavares Ribeiro

**Abstract**

In [Abr97], Abrashkin established the Brückner-Vostokov formula for the Hilbert symbol of a formal group under the assumption that roots of unity belong to the base field. The main motivation of this work is to remove this hypothesis. It is obtained by combining methods of $(\varphi, \Gamma)$-modules and a cohomological interpretation of Abrashkin's technique. To do this, we build $(\varphi, \Gamma)$-modules adapted to the false Tate curve extension and generalize some related tools like the Herr complex with explicit formulas for the cup-product and the Kummer map.

## Contents



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# Introduction

## 0.1 $(\varphi, \Gamma)$-modules

Let $p$ be a prime number and $K$ a finite extension of $\mathbb{Q}_p$ with residue field $k$. Fix $\overline{K}$ an algebraic closure of $K$ and note $G_K = \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of $K$. Let us furthermore introduce $K_\infty = \cup_n K(\zeta_{p^n})$ the cyclotomic extension of $K$ and $\Gamma_K = \mathrm{Gal}(K_\infty/K)$.
The context of this work is the theory of $p$-adic representations of the Galois group of a local field, here $G_K$. We are particularly interested in $\mathbb{Z}_p$-adic representations of $G_K$, *i.e.* $\mathbb{Z}_p$-modules of finite type endowed with a linear and continuous action of $G_K$.
In [Fon90], Fontaine introduced the notion of a $(\varphi, \Gamma_K)$-module over the ring $\mathbf{A}_K$. This ring is, when $K$ is absolutely unramified, the set of power series $\sum_{n\in\mathbb{Z}} a_n X^n$

with $a_n \in \mathcal{O}_K$, $a_n$ $p$-adicly converging to 0 as $n$ goes to $-\infty$ and $X$ a variable on which $\varphi$ and $\Gamma_K$ act for $\gamma \in \Gamma_K$ via

$$\varphi(X) = (1+X)^p - 1 \ ; \quad \gamma(X) = (1+X)^{\chi(\gamma)} - 1$$

where $\chi$ is the cyclotomic character.
A $(\varphi, \Gamma_K)$-module over $\mathbf{A}_K$ is then a module of finite type over $\mathbf{A}_K$ endowed with commuting semi-linear actions of $\varphi$ and $\Gamma_K$.
Fontaine defined an equivalence of categories between the category of $\mathbb{Z}_p$-adic representations of $G_K$ and the category of étale $(\varphi, \Gamma_K)$-modules over $\mathbf{A}_K$. Cherbonnier and Colmez showed in [CC98] that any $p$-adic representation is overconvergent, which established a first link between the $(\varphi, \Gamma_K)$-module $D(V)$ of a representation $V$ and its de Rham module which contains the geometric information on $V$. Berger then, in [Ber02], showed how to recover the de Rham module $D_{dR}(V)$, the semi-stable module $D_{st}(V)$ or the crystalline module $D_{crys}(V)$ of Fontaine's theory from $D(V)$. For absolutely unramified crystalline representations, Wach furnished in [Wac96] another powerful construction which permits to recover $D_{crys}(V)$ in the $(\varphi, \Gamma_K)$-module $D(V)$. This construction was studied in details and made more precise by Berger ([Ber04]). $(\varphi, \Gamma_K)$-modules are also intimately linked to Iwasawa theory as was shown in works by Cherbonnier and Colmez ([CC99]), Benois ([Ben00]) or Berger ([Ber03]).
Let us eventually cite another significant result brought by Herr in his PhD thesis ([Her98]) who furnished a three terms complex in the $(\varphi, \Gamma_K)$-module of a representation, whose homology computes the Galois cohomology of the representation.

## 0.2 The false Tate curve extension

The construction of $(\varphi, \Gamma_K)$-modules lies on the use of the cyclotomic tower and shows its fundamental role in the study of $p$-adic representations. But another extension appears as particularly significant.
Fix $\pi$ a uniformizer of $K$ and $\pi_n$ a system of $p^n$-th roots of $\pi$:

$$\pi_0 = \pi \ \text{ et } \ \forall n \in \mathbb{N}, \ \pi_{n+1}^p = \pi_n.$$

It is then the behavior in extension $K_\pi = \cup_n K(\pi_n)$ which makes the difference between a crystalline and a semi-stable representation.
Let us cite moreover the following remarkable result.

**Theorem 0.1. (Breuil, Kisin)**
*The forgetful functor from the category of $p$-adic crystalline representations of $G_K$ to the category of $p$-adic representations of $G_{K_\pi}$ is fully faithful.*

This theorem was conjectured by Breuil in [Bre99] where it was shown under some conditions on the Hodge-Tate weights of the representation, with the help of objects

very similar to Fontaine's $(\varphi, \Gamma_K)$-modules. Kisin proved this result unconditionally in [Kis06]. Other results, in particular by Abrashkin ([Abr97, Abr95]), encourage us to introduce, like Breuil, $(\varphi, \Gamma)$-modules where the cyclotomic extension $K_\infty$ is replaced by $K_\pi$. However $K_\pi/K$ is not Galois and we only get $\varphi$-modules (also studied by Fontaine in [Fon90]).
Let us then consider the Galois closure $L$ of $K_\pi$ which is nothing more than the compositum of $K_\pi$ and $K_\infty$, a metabelian extension of $K$, *the false Tate curve extension.* What we lose here is the explicit description of the field of norms of this extension. Note $G_\infty = \mathrm{Gal}(L/K)$. Our first result can then, for $\mathbf{A}' = \mathbf{A}$ or $\tilde{\mathbf{A}}$, and $\mathbf{A}'_L = \mathbf{A}'^{G_L}$ (where $\mathbf{A}$ are $\tilde{\mathbf{A}}$ Fontaine rings defined in Paragraph 1.2), be expressed as:

**Theorem 0.2.**
*The functor*

$$\begin{array}{ccc} \{\mathbb{Z}_p - \textit{adic representations of } G_K\} & \to & \{\textit{étale } (\varphi, G_\infty) - \textit{modules over } \mathbf{A}'_L\} \\ V & \mapsto & D_L(V) = (V \otimes_{\mathbb{Z}_p} \mathbf{A}')^{G_L} \end{array}$$

*is an equivalence of categories.*

In fact we show that the $(\varphi, G_\infty)$-module $D_L(V)$ is nothing but the scalar extension of the usual $(\varphi, \Gamma_K)$-module $D(V)$ from $\mathbf{A}_K$ to $\mathbf{A}'_L$.

## 0.3 Galois cohomology

We are now able to associate with a representation a $(\varphi, G_\infty)$-module giving a better control of the behavior of the representation in the extension $K_\pi$. But we would like to use tools available in the classical framework, first of all Herr's complex. Recall that in the usual case of $(\varphi, \Gamma_K)$-modules, Herr showed in [Her98] that the homology of the complex

$$0 \longrightarrow D(V) \xrightarrow{f_1} D(V) \oplus D(V) \xrightarrow{f_2} D(V) \longrightarrow 0$$

with maps

$$f_1 = \begin{pmatrix} \varphi - 1 \\ \gamma - 1 \end{pmatrix} \text{ et } f_2 = (\gamma - 1, 1 - \varphi)$$

computes the Galois cohomology of the representation $V$.
Since the group $G_\infty$ is now of dimension 2, the corresponding complex loses some simplicity. Let $\tau$ be a topological generator of the sub-group $\mathrm{Gal}(L/K_\infty)$ and $\gamma$ a topological generator of $\mathrm{Gal}(L/K_\pi)$ satisfying $\gamma\tau\gamma^{-1} = \tau^{\chi(\gamma)}$, it can be described as:

**Theorem 0.3.**
*Let $V$ be a $\mathbb{Z}_p$-adic representation of $G_K$ and $D$ its $(\varphi, G_\infty)$-module. The homology of the complex*

$$0 \longrightarrow D \xrightarrow{\alpha} D \oplus D \oplus D \xrightarrow{\beta} D \oplus D \oplus D \xrightarrow{\eta} D \longrightarrow 0$$

*where*

$$\alpha = \begin{pmatrix} \varphi - 1 \\ \gamma - 1 \\ \tau - 1 \end{pmatrix}, \beta = \begin{pmatrix} \gamma - 1 & 1 - \varphi & 0 \\ \tau - 1 & 0 & 1 - \varphi \\ 0 & \tau^{\chi(\gamma)} - 1 & \delta - \gamma \end{pmatrix}, \eta = \begin{pmatrix} \tau^{\chi(\gamma)} - 1, & \delta - \gamma, & \varphi - 1 \end{pmatrix}$$

*with* $\delta = (\tau^{\chi(\gamma)} - 1)(\tau - 1)^{-1} \in \mathbb{Z}_p[[\tau - 1]]$, *identifies canonically and functorially with the continuous Galois cohomology of* $V$.

In fact, we get explicit isomorphisms. In particular for the first cohomology group, let $(x, y, z) \in \ker \beta$, let $b$ be a solution in $V \otimes \mathbf{A}'$ of

$$(\varphi - 1)b = x,$$

then the above theorem associates with the class of the triple $(x, y, z)$ the class of the cocycle:

$$c \ : \ \sigma \mapsto c_\sigma = -(\sigma - 1)b + \gamma^n \frac{\tau^m - 1}{\tau - 1} z + \frac{\gamma^n - 1}{\gamma - 1} y$$

where $\sigma_{|G_\infty} = \gamma^n \tau^m$.
Moreover, like Herr in [Her01], we furnish explicit formulas describing the cup-product in terms of the four terms Herr complex above.

## 0.4 Explicit formulas for the Hilbert symbol

The Hilbert symbol, for a field $K$ containing the group $\mu_{p^n}$ of $p^n$-th roots of unity is defined as the pairing

$$\begin{aligned} (,)_{p^n} : K^*/K^{*p^n} \times K^*/K^{*p^n} & \rightarrow \mu_{p^n} \\ (a, b)_{p^n} & = \left(\sqrt[p^n]{b}\right)^{r_K(a) - 1} \end{aligned}$$

where $r_K : K^* \to G_K^{\mathrm{ab}}$ is the reciprocity map.
Since 1858 and Kummer's work, many explicit formulas have been given for the Hilbert symbol. Let us cite the one of Coleman ([Col81]): suppose that $K = K_0(\zeta_{p^n})$ where $K_0$ is a finite unramified extension of $\mathbb{Q}_p$ and $\zeta_{p^n}$ a fixed primitive $p^n$-th root of unity. Note $W$ the ring of integers of $K_0$. If $F \in 1 + (p, X) \subset W[[X]]$, then $F(\zeta_{p^n} - 1)$ is a principal unit in $K$ and all of them are obtained in that way. Extend the absolute Frobenius $\varphi$ from $W$ to $W[[X]]$ by putting $\varphi(X) = (1+X)^p - 1$. Denote for $F \in W[[X]]$

$$\mathscr{L}(F) = \frac{1}{p} \log \frac{F(X)^p}{\varphi(F(X))} \in W[[X]].$$

Then for $F \in 1 + (p, X)$,

$$\mathscr{L}(F) = \left(1 - \frac{\varphi}{p}\right) \log F(X).$$

Coleman's formula can then be written as:

**Theorem 0.4. (Coleman)**
*Let* $F, G \in 1 + (p, X) \subset W[[X]]$, *then*

$$(F(\zeta_{p^n} - 1), G(\zeta_{p^n} - 1))_{p^n} = \zeta_{p^n}^{[F,G]_n}$$

*where*

$$[F, G]_n = \mathrm{Tr}_{K_0/\mathbb{Q}_p} \circ \mathrm{Res}_X \frac{1}{\varphi^n(X)} \left( \mathscr{L}(G) d \log F - \frac{1}{p} \mathscr{L}(F) d \log G^{\varphi} \right).$$

Let us furthermore cite the Brückner-Vostokov formula: suppose now that $p \neq 2$, let $\zeta_{p^n} \in K$, let $W$ be the ring of integers of $K_0$, the maximal unramified extension of $K/\mathbb{Q}_p$. Extend the Frobenius $\varphi$ from $W$ to $W[[Y]][1/Y]$ via $\varphi(Y) = Y^p$. Fix moreover $\pi$ a uniformizer of $K$.

**Theorem 0.5. (Brückner-Vostokov)**
*Let* $F, G \in (W[[Y]][1/Y])^{\times}$, *then*

$$(F(\pi), G(\pi))_{p^n} = \zeta_{p^n}^{[F,G]_n}$$

*where*

$$[F, G]_n = \mathrm{Tr}_{K_0/\mathbb{Q}_p} \circ \mathrm{Res}_Y \frac{1}{s^{p^n} - 1} \left( \mathscr{L}(G) d \log F - \frac{1}{p} \mathscr{L}(F) d \log G^{\varphi} \right)$$

*with* $s \in W[[Y]]$ *such that* $s(\pi) = \zeta_{p^n}$.

The purpose of the second part of this work is to show a generalization of this formula to the case of formal groups.
Remark that there are other types of formulas, in particular the one of Sen ([Sen80]), generalized to formal groups by Benois in [Ben97].
We refer interested readers to Vostokov's [Vos00] which provides a comprehensive background on explicit formulas for the Hilbert symbol.

## 0.5 An explicit formula for formal groups

Let $G$ be a connected smooth formal group of dimension $d$ and of finite height $h$ over the ring of Witt vectors $W = W(k)$ with coefficients in a finite field $k$. Let $K_0$ be the fraction field of $W$ and $K$ a finite extension of $K_0$ containing the $p^M$-torsion $G[p^M]$ of $G$. Define then the Hilbert symbol of $G$ to be the pairing

$$\begin{array}{rcl} (,)_{G,M} : K^* \times G(\mathfrak{m}_K) & \rightarrow & G[p^M] \\ (x, \beta)_{G,M} & = & r_K(x)(\beta_1) -_G \beta_1 \end{array}$$

where $r_K : K^* \rightarrow G_K^{\mathrm{ab}}$ is the reciprocity map and $\beta_1$ satisfies

$$p^M \mathrm{id}_G \beta_1 = \beta.$$

Fix a basis of logarithms of $G$ under the form of a vectorial logarithm $l_G \in K_0[[\mathbf{X}]]^d$ where $\mathbf{X} = (X_1, \dots, X_d)$ such that one has the formal identity

$$l_G(\mathbf{X} +_G \mathbf{Y}) = l_G(\mathbf{X}) + l_G(\mathbf{Y}).$$

Complete $l_G$ with almost-logarithms $m_G \in K_0[[\mathbf{X}]]^{h-d}$ in a basis $\begin{pmatrix} l_G \\ m_G \end{pmatrix}$ of the Dieudonné module of $G$.

Fontaine defined in [Fon77] (see also [Col92] for an explicit description) a pairing between the Dieudonné module and the Tate module of $G$

$$T(G) = \varprojlim G[p^n].$$

Honda showed in [Hon70] the existence of a formal power series of the form $\mathcal{A} = \sum_{n\geq 1} F_n \varphi^n$ with $F_n \in M_d(W)$ such that

$$\left(1 - \frac{\mathcal{A}}{p}\right) \circ l_G(\mathbf{X}) \in M_d(W[[\mathbf{X}]]).$$

Let us introduce moreover the approximated period matrix. Fix $(o^1, \dots, o^h)$ a basis of $T(G)$ where $o^i = (o^i_n)_{n\geq 1}$ such that $p\mathrm{id}_G o^i_n = o^i_{n-1}$. Approach $(o^1 = (o^1_n)_n, \dots, o^h)$ by a basis $(o^1_M, \dots, o^h_M)$ of $G[p^M]$. Then for all $i$, choose $\hat{o}^i_M \in F(YW[[Y]])$ such that $\hat{o}^i_M(\pi) = o^i_M$. The matrix $\mathcal{V}_Y$ is then

$$\mathcal{V}_Y = \begin{pmatrix} p^M l_G(\hat{o}^1_M) & \dots & p^M l_G(\hat{o}^h_M) \\ p^M m_G(\hat{o}^1_M) & \dots & p^M m_G(\hat{o}^h_M) \end{pmatrix}.$$

It is an approximation of the period matrix $\mathcal{V}$.

Now we can state the reciprocity law which generalizes the Brückner-Vostokov law and which constitutes the goal of the second part of this work:

**Theorem 0.6.**
*Let $\alpha \in (W[[Y]][\frac{1}{Y}])^\times$ and $\beta \in G(YW[[Y]])$. Coordinates of the Hilbert symbol $(\alpha(\pi), \beta(\pi))_{G,M}$ in the basis $(o^1_M, \dots, o^h_M)$ are*

$$(\mathrm{Tr}_{W/\mathbb{Z}_p} \circ \mathrm{Res}_Y)\mathcal{V}_Y^{-1} \left( \begin{pmatrix} (1 - \frac{\mathcal{A}}{p}) \circ l_G(\beta) \\ 0 \end{pmatrix} d_{\log}\alpha(Y) - \mathscr{L}(\alpha) \frac{d}{dY} \begin{pmatrix} \frac{\mathcal{A}}{p} \circ l_G(\beta) \\ m_G(\beta) \end{pmatrix} \right).$$

This formula was shown by Abrashkin in [Abr97] under the assumption that $K$ contains $p^M$-th roots of unity. Vostokov and Demchenko proved it in [VD00] without any condition on $K$ for formal groups of dimension 1.

## 0.6 The strategy

The main idea of the proof is due to Benois who carried it out in [Ben00] to show Coleman's reciprocity law. Let us recall what it consists in.
The Hilbert symbol can be seen as a cup-product via the following commutative diagram

$$\begin{tikzcd} K^* \times K^* \arrow[r, "(,)_{p^n}"] \arrow[d, "\kappa\times\kappa"'] & \mu_{p^n} \\ H^1(K,\mu_{p^n}) \times H^1(K,\mu_{p^n}) \arrow[r, "\cup"] & H^2(K,\mu_{p^n}^{\otimes 2}) \arrow[u, "\mathrm{inv}_K"'] \end{tikzcd}$$

where $\kappa$ is Kummer's map. He first explicitly computed Kummer's map in terms of the Herr complex associated with the representation $\mathbb{Z}_p(1)$, then he used Herr's cup-product explicit formulas and he finally computed the image of the couple he obtained via the isomorphism $\mathrm{inv}_K$.
For a formal group, the situation is rather similar, we get the diagram

$$\begin{tikzcd} K^* \times G(\mathfrak{m}_K) \arrow[r, "(,)_{G,M}"] \arrow[d, "\kappa\times\kappa_G"'] & G[p^M] \\ H^1(K,\mu_{p^M}) \times H^1(K,G[p^M]) \arrow[r, "\cup"] & H^2(K,\mu_{p^M}\otimes G[p^M]) \arrow[u, "\mathrm{inv}_K"'] \end{tikzcd}$$

with

$$G[p^M] \simeq (\mathbb{Z}/p^M\mathbb{Z})^h,$$
$$\text{and} \quad H^2(K,\mu_{p^M}\otimes G[p^M]) \simeq H^2(K,\mathbb{Z}/p^M\mathbb{Z}(1)) \otimes_{\mathbb{Z}/p^M\mathbb{Z}} G[p^M].$$

Formulas for the Kummer map and the cup-product are shown in the section on $(\varphi,\Gamma)$-modules. The computation of the explicit formula for the map $\kappa_G : G(\mathfrak{m}_K) \to H^1(K,G[p^M])$ constitutes the technical axis of this work.
Abrashkin made use of the Witt symbol, and to conclude via the field of norms of extension $K_\pi/K$, he used the compatibility of the reciprocity map between the field of norms of an extension and the basis field. Some of his intermediate results ([Abr97, Propositions 3.7 and 3.8]) can be directly translated in the language of $(\varphi,G_\infty)$-modules. Indeed, we want to compute a triple $(x,y,z)$ in the first homology group of the Herr generalized complex associated with the representation $G[p^M]$. Abrashkin's results permit to obtain $x$, the vanishing of $y$ and the belonging of $z$ to $W(\mathfrak{m}_{\tilde{\mathbf{E}}})$ (where $\tilde{\mathbf{E}}$ is a Fontaine ring, cf. 1.2 below). However we need to know $z$ modulo $XW(\mathfrak{m}_{\tilde{\mathbf{E}}})$ and then we have to carry Abrashkin's computations to the higher order.

## 0.7 Organization of the paper

This work splits in two parts. In the first one, we introduce $(\varphi,G_\infty)$-modules and give the associated Herr complex with explicit formulas between its homology and

the cohomology of the representation. Then we provide explicit formulas for the cup-product and the Kummer map in terms of the Herr complex.
The second part is devoted to the proof of the Brückner-Vostokov formula for formal groups. The main difficulty lie in the fact that the period matrix does not live in the right place: we introduce an approximated period matrix and show that it enjoys similar properties as the original matrix modulo suitable rings. Then, we carry out the computation of the Hilbert symbol in terms of the Herr complex.

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# 1 $(\varphi, \Gamma)$-modules and cohomology

## 1.1 Notation

Let $p$ be a prime.
Let us recall (cf. [Ser68]) that if $\mathbb{K}$ is a perfect field of characteristic $p$, one can endow the space $\mathbb{K}^{\mathbb{N}}$ of sequences of elements in $\mathbb{K}$ with a structure of a local ring of characteristic 0 absolutely unramified and with residue field $\mathbb{K}$. It is called the ring of Witt vectors over $\mathbb{K}$ and is denoted by $W(\mathbb{K})$. Recall moreover that this construction permits to define a multiplicative section of the canonical surjection

$$W(\mathbb{K}) \to \mathbb{K},$$

called the Teichmüller representative and denoted by $[\ ]$. If $R$ is a (unitary or not) subring of $\mathbb{K}$, we still denote by $W(R)$ the Witt vectors with coefficients in $R$. It is then a subring of $W(\mathbb{K})$.

Fix $K$ a finite extension of $\mathbb{Q}_p$ with residue field $k$.
Denote $W = W(k)$ the ring of Witt vectors over $k$. Then $K_0 = W \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ identifies with the maximal unramified sub-extension of $\mathbb{Q}_p$ in $K$.
Fix $\overline{K}$ an algebraic closure of $K$ and denote

$$G_K = \mathrm{Gal}(\overline{K}/K)$$

the absolute Galois group of $K$ and $\mathbb{C}_p$ the $p$-adic completion of $\overline{K}$. Endow $\mathbb{C}_p$ with the $p$-adic valuation $v_p$ normalized by

$$v_p(p) = 1.$$

Recall that the action of $G_K$ on $\overline{K}$ extends by continuity to $\mathbb{C}_p$.
Let us fix $\varepsilon = (\zeta_{p^n})_{n\geq 0}$ a coherent system of $p^n$-th roots of unity, *i.e.* $\zeta_{p^n}^p = \zeta_{p^{n-1}}$ for all $n$, $\zeta_1 = 1$ and $\zeta_p \neq 1$. Then

$$K_\infty := \bigcup_{n\in\mathbb{N}} K(\zeta_{p^n})$$

is the cyclotomic extension of $K$. Denote $G_{K_\infty} = \mathrm{Gal}(\overline{K}/K_\infty)$ its absolute Galois group and $\Gamma_K = \mathrm{Gal}(K_\infty/K)$ the quotient.
Let us fix as well $\pi$ a uniformizer of $K$ and $\rho = (\pi_{p^n})_{n\geq 0}$ a coherent system of $p^n$-th roots of $\pi$. Denote

$$K_\pi = \bigcup_{n\geq 0} K(\pi_{p^n}).$$

The extension $K_\pi/K$ is not Galois, so put

$$L = \bigcup_{n\geq 0} K(\zeta_{p^n}, \pi_{p^n})$$

its Galois closure. It is the compositum of $K_\pi$ and $K_\infty$. Denote $G_L = \mathrm{Gal}(\overline{K}/L)$ its absolute Galois group and $G_\infty = \mathrm{Gal}(L/K)$ the quotient. The cyclotomic character $\chi : G_K \to \mathbb{Z}_p^*$ factorizes through $G_\infty$ (even through $\Gamma_K$) ; it is also true for the map $\psi : G_K \to \mathbb{Z}_p$ defined by

$$\forall g \in G_K \quad g(\pi_{p^n}) = \pi_{p^n}\zeta_{p^n}^{\psi(g)}.$$

Moreover, the group $G_\infty$ identifies with the semi-direct product $\mathbb{Z}_p \rtimes \Gamma_K$. So $G_\infty$ is topologically generated by two elements, $\gamma$ and $\tau$ satisfying:

$$\gamma\tau\gamma^{-1} = \tau^{\chi(\gamma)}.$$

Let us fix $\gamma$ and choose $\tau$ such that $\psi(\tau) = 1$, *i.e.* with

$$\tau(\rho) = \rho\varepsilon.$$

We adopt the convention that complexes have their first term in degree $-1$ if this term is 0, and otherwise in degree 0.

**Remark** The group $G_\infty$ is a $p$-adic Lie group so that the extension $L/K$ is arithmetically profinite (cf [Win83, Ven03]).

## 1.2 The field $\tilde{\mathbf{E}}$, the ring $\tilde{\mathbf{A}}$ and some of their subrings.

We refer to [Fon90] for results of this section. However we adopt Colmez' notation. Rings $R$, $W(\mathrm{Frac}R)$ and $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}$ of [Fon90] become $\tilde{\mathbf{E}}^+$, $\tilde{\mathbf{A}}$ and $\mathbf{A}$.

Define $\tilde{\mathbf{E}}$ as the inverse limit

$$\tilde{\mathbf{E}} = \lim_{\substack{\longleftarrow \\ n}} \mathbb{C}_p$$

where transition maps are exponentiation to the power $p$. An element of $\tilde{\mathbf{E}}$ is then a sequence $x = (x^{(n)})_{n\in\mathbb{N}}$ satisfying

$$(x^{(n+1)})^p = x^{(n)} \ \forall n \in \mathbb{N}.$$

Endow $\tilde{\mathbf{E}}$ with the addition

$$x + y = s \text{ where } s^{(n)} = \lim_{m\to+\infty}(x^{(n+m)} + y^{(n+m)})^{p^m}$$

and the product

$$x.y = t \text{ where } t^{(n)} = x^{(n)}.y^{(n)}.$$

These operations make $\tilde{\mathbf{E}}$ into a field of characteristic $p$, algebraically closed and complete for the valuation

$$v_{\mathbf{E}}(x) := v_p(x^{(0)}).$$

The ring of integers of $\tilde{\mathbf{E}}$, denoted by $\tilde{\mathbf{E}}^+$, identifies then with the inverse limit $\varprojlim \mathcal{O}_{\mathbb{C}_p}$. It is a local ring whose maximal ideal, denoted by $\mathfrak{m}_{\tilde{\mathbf{E}}}$ identifies with $\varprojlim \mathfrak{m}_{\mathbb{C}_p}$ and whose residue field is isomorphic to $\overline{k}$.
The field $\tilde{\mathbf{E}}$, as well as its ring of integers $\tilde{\mathbf{E}}^+$, still has a natural action of $G_K$ which is continuous with respect to the $v_{\mathbf{E}}$-adic topology. Define the Frobenius

$$\varphi \ : \ x \mapsto x^p$$

which acts continuously, commutes with the action of $G_K$ and stabilizes $\tilde{\mathbf{E}}^+$.

Let $\tilde{\mathbf{A}} = W(\tilde{\mathbf{E}})$ be the ring of Witt vectors on $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{A}}^+ = W(\tilde{\mathbf{E}}^+)$.
Any element of $\tilde{\mathbf{A}}$ (respectively $\tilde{\mathbf{A}}^+$) can be written uniquely as

$$\sum_{n\in\mathbb{N}} p^n[x_n]$$

where $(x_n)_{n\in\mathbb{N}}$ is a sequence of elements in $\tilde{\mathbf{E}}$ (respectively in $\tilde{\mathbf{E}}^+$).
The topology on $\tilde{\mathbf{A}}$ comes from the product topology on $W(\tilde{\mathbf{E}}) = \tilde{\mathbf{E}}^{\mathbb{N}}$. This topology is compatible with the ring structure on $\tilde{\mathbf{A}}$. It is weaker than the $p$-adic topology.
Let us remark that the sequences $\varepsilon$ and $\rho$ introduced below define elements in $\tilde{\mathbf{E}}^+$. Denote

$$X = [\varepsilon] - 1 \text{ and } Y = [\rho].$$

These are elements of $\tilde{\mathbf{A}}^+$ and even of $W(\mathfrak{m}_{\tilde{\mathbf{E}}})$. They are topologically nilpotent. We also have a basis of neighborhoods of 0 in $\tilde{\mathbf{A}}$:

$$\{p^n\tilde{\mathbf{A}} + X^m\tilde{\mathbf{A}}^+\}_{(n,m)\in\mathbb{N}^2} \text{ and } \{p^n\tilde{\mathbf{A}} + Y^m\tilde{\mathbf{A}}^+\}_{(n,m)\in\mathbb{N}^2}.$$

Let $W[[X,Y]]$ denote the subring of $\tilde{\mathbf{A}}^+$ consisting in sequences in $X$ and $Y$ ; it is stable under the action of $G_K$ which is given by:

$$g(1+X) = (1+X)^{\chi(g)} \text{ and } g(Y) = Y(1+X)^{\psi(g)}$$

and the one of $\varphi$:

$$\varphi(X) = (1+X)^p - 1 \text{ and } \varphi(Y) = Y^p.$$

**Remark**

The specialization morphism for polynomials

$$\begin{array}{ccc} W[X_1, X_2] & \to & \tilde{\mathbf{A}}^+ \\ X_1,\ X_2 & \mapsto & X,\ Y \end{array}$$

is injective. However, the one for formal power series

$$\begin{array}{ccc} W[[X_1, X_2]] & \to & \tilde{\mathbf{A}}^+ \\ X_1,\ X_2 & \mapsto & X,\ Y \end{array}$$

is not a priori.

Let $\mathbf{A}_{\mathbb{Q}_p}$ denote the $p$-adic completion of $\mathbb{Z}_p[[X]][\frac{1}{X}]$, it consists in the set

$$\mathbf{A}_{\mathbb{Q}_p} = \left\{ \sum_{n\in\mathbb{Z}} a_n X^n \mid \forall n \in \mathbb{Z},\ a_n \in \mathbb{Z}_p \text{ and } a_n \underset{n\to-\infty}{\longrightarrow} 0 \right\}.$$

It is a local $p$-adic, complete subring of $\tilde{\mathbf{A}}$, with residue field $\mathbb{F}_p((\varepsilon-1))$. Define $\mathbf{A}$ the $p$-adic completion of the maximal unramified extension of $\mathbf{A}_{\mathbb{Q}_p}$ in $\tilde{\mathbf{A}}$. Its residue field is then the separable closure of $\mathbb{F}_p((\varepsilon-1))$ in $\tilde{\mathbf{E}}$. Denote this field by $\mathbf{E}$. It is a dense subfield of $\tilde{\mathbf{E}}$.

## 1.3 Rings of $p$-adic periods.

### 1.3.1 $B_{dR}$ and some of its subrings

We refer to [Fon94] for further details on these rings.

The map

$$\theta : \left\{ \begin{array}{ccc} \tilde{\mathbf{A}}^+ & \to & \mathcal{O}_{\mathbb{C}_p} \\ \sum_{n\geq 0} p^n [r_n] & \mapsto & \sum_{n\geq 0} p^n r_n^{(0)} \end{array} \right.$$

is surjective, with kernel $W^1(\tilde{\mathbf{E}}^+)$ which is a principal ideal of $\tilde{\mathbf{A}}^+$ generated, for instance, by $\omega = X/\varphi^{-1}(X)$. Denote

$$B_{dR}^+ = \varprojlim_n (\tilde{\mathbf{A}}^+ \otimes \mathbb{Q}_p)/(W^1(\tilde{\mathbf{E}}^+) \otimes \mathbb{Q}_p)^n$$

the completion of $\tilde{\mathbf{A}}^+ \otimes \mathbb{Q}_p$ with respect to the $W^1(\tilde{\mathbf{E}}^+)$-adic topology. The action of $G_K$ on $\tilde{\mathbf{A}}^+$ extends by continuity to $B_{dR}^+$. Yet it is not the case of the Frobenius $\varphi$ which is not continuous with respect to the $W^1(\tilde{\mathbf{E}}^+)$-adic topology. The sequence

$$\log[\varepsilon] = \sum_{n\geq 1} (-1)^{n+1} \frac{X^n}{n}$$

converges in $B_{dR}^+$ towards an element denoted by $t$. Define then

$$B_{dR} = B_{dR}^+[1/t].$$

It is the fraction field of $B_{dR}^+$. It is still endowed with an action of $G_K$ for which

$$B_{dR}^{G_K} = K$$

and with a compatible, decreasing, exhaustive filtration

$$\mathrm{Fil}^k B_{dR} = t^k B_{dR}^+.$$

Define now the ring $A_{crys}$ to be the $p$-adic completion of the divided powers envelop of $\tilde{\mathbf{A}}^+$ with respect to $W^1(\tilde{\mathbf{E}}^+)$. It consists in the sequences

$$\left\{ \sum_{n\geq 0} a_n \frac{\omega^n}{n!} \text{ such that } a_n \in \tilde{\mathbf{A}}^+ \text{ and } a_n \to 0 \text{ } p\text{-adically.} \right\}.$$

This ring is naturally a subring of $B_{dR}$. Moreover, the sequence defining $t$ still converges in $A_{crys}$ and we set

$$B_{crys}^+ = A_{crys} \otimes \mathbb{Q}_p \text{ and } B_{crys} = B_{crys}^+[1/t] = A_{crys}[1/t].$$

Moreover, if one chooses $\tilde{p} = (p_0, p_1, \dots) \in \tilde{\mathbf{E}}$ with $p_0 = p$, then the series $\log \frac{[\tilde{p}]}{p}$ converges in $B_{dR}$ towards a limit denoted by $\log[\tilde{p}]$ (with the implicit convention $\log p = 0$). Define then

$$B_{st} = B_{crys}[\log[\tilde{p}]].$$

It is still a subring of $B_{dR}$.
All these rings, endowed with their $p$-adic topology, come with a continuous action of $G_K$, the filtration induced by the one on $B_{dR}$, and a Frobenius $\varphi$ extending by continuity the one on $\tilde{\mathbf{A}}^+$. Note that

$$B_{crys}^{G_K} = K_0 \quad \text{and} \quad B_{st}^{G_K} = K_0.$$

### 1.3.2 A classification of $G_K$-representations

We call a *$\mathbb{Z}_p$-adic representation of $G_K$* any finitely generated $\mathbb{Z}_p$-module with a linear, continuous action of $G_K$ and a *$p$-adic representation of $G_K$* any finite dimensional $\mathbb{Q}_p$-vector space with a linear, continuous action of $G_K$. A $\mathbb{Z}_p$-adic representation is then turned into a $p$-adic representation by tensorizing by $\mathbb{Q}_p$.
Let $V$ be a $p$-adic representation of $G_K$. Note

$$\begin{aligned} D_{dR}(V) &:= (V \otimes_{\mathbb{Q}_p} B_{dR})^{G_K} \\ D_{st}(V) &:= (V \otimes_{\mathbb{Q}_p} B_{st})^{G_K} \\ D_{crys}(V) &:= (V \otimes_{\mathbb{Q}_p} B_{crys})^{G_K}. \end{aligned}$$

$D_{dR}(V)$ (respectively $D_{st}(V)$, $D_{crys}(V)$) is a $K$ (respectively $K_0$, $K_0$)-vector space of dimension lower or equal to the dimension of $V$ on $\mathbb{Q}_p$. The representation $V$ is said to be *de Rham* (respectively *semi-stable*, *crystalline*) when these dimensions are equal.
One immediately sees that crystalline representations are semi-stable and semi-stable representations are de Rham.
We say as well that a $\mathbb{Z}_p$-adic representation $V$, free over $\mathbb{Z}_p$, is de Rham, semi-stable or crystalline when so is the $p$-adic representation $V \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

**Example** *The false Tate curve*
Let us define the false Tate curve (or Tate's representation) by

$$V_{Tate} = \mathbb{Z}_p e_1 + \mathbb{Z}_p e_2$$

with the action of $G_K$:

$$\begin{cases} g(e_1) = \chi(g) e_1 \\ g(e_2) = \psi(g) e_1 + e_2 \end{cases}$$

for all $g \in G_K$, where $\chi$ is the cyclotomic character and $\psi$ is defined in Paragraph 1.1. This representation is an archetypic semi-stable representation and will be an important reference. We will confront our approach with it, in particular modified $(\varphi, \Gamma)$-modules. For the moment, just note that the action of $G_K$ on $V_{Tate}$ factorizes through $G_\infty$.
The name "false Tate curve" comes from the similarity of this module with the Tate module of an elliptic curve with split multiplicative reduction at $p$.

## 1.4 Fontaine's theory

Let $R$ be a topological ring with a linear, continuous action of some group $\Gamma$ and a continuous Frobenius $\varphi$ commuting with the action of $\Gamma$. Call a *$(\varphi, \Gamma)$-module on $R$* any finitely generated $R$-module $M$ with commuting semi-linear actions of $\Gamma$ and $\varphi$.

A $(\varphi,\Gamma)$-module on $R$ is moreover said *étale* if the image of $\varphi$ generates $M$ as an $R$-module:

$$R\varphi(M) = M.$$

### 1.4.1 The classical case

Let us recall the theory of $(\varphi,\Gamma)$-modules introduced by Fontaine in [Fon90].

Set $\mathbf{A}_K = \mathbf{A}^{G_{K_\infty}}$.
Define the functors

$$D : V \mapsto D(V) = (\mathbf{A} \otimes_{\mathbb{Z}_p} V)^{G_{K_\infty}}$$

from the category of $\mathbb{Z}_p$-adic representations of $G_K$ to the one of $(\varphi,\Gamma_K)$-modules on $\mathbf{A}_K$ and

$$V : M \mapsto V(M) = (\mathbf{A} \otimes_{\mathbf{A}_K} M)^{\varphi=1}$$

from the category of étale $(\varphi,\Gamma_K)$-modules on $\mathbf{A}_K$ to the one of $\mathbb{Z}_p$-adic representations of $G_K$. The following theorem was shown by Fontaine ([Fon90]):

**Theorem 1.1.**
*Natural maps*

$$\mathbf{A} \otimes_{\mathbf{A}_K} D(V) \to \mathbf{A} \otimes_{\mathbb{Z}_p} V$$
$$\mathbf{A} \otimes_{\mathbb{Z}_p} V(M) \to \mathbf{A} \otimes_{\mathbf{A}_K} M$$

*are isomorphisms. In particular, $D$ and $V$ are quasi-inverse equivalences of categories between the category of $\mathbb{Z}_p$-adic representations of $G_K$ and the one of étale $(\varphi,\Gamma_K)$-modules on $\mathbf{A}_K$.*

**Example** The $(\varphi,\Gamma_K)$-module of the false Tate curve admits a basis of the form $(1\otimes e_1, b\otimes e_1 + 1\otimes e_2)$ where $b \in \mathbf{A}_L$ satisfies $(\tau-1)b = -1$. However $V_{Tate}$ is not potentially crystalline, and then, by a theorem of Wach (cf. [Wac96]), not *of finite height*, which means $b \notin \mathbf{A}_L^+ = \mathbf{A}_L \bigcap \tilde{\mathbf{A}}^+$.

We want to build a $(\varphi,\Gamma)$-module which furnishes more information (which will then be redundant but easier to use) on the behavior of the associated representation in the extension $K_\pi/K$ or in its Galois closure $L/K$. For this, we want $\Gamma = G_\infty$.

### 1.4.2 The metabelian case

Suppose $\mathbf{A}' = \mathbf{A}$ or $\mathbf{A}' = \tilde{\mathbf{A}}$. Then, $\mathbf{A}'$ is a complete $p$-adic valuation ring, stable under both $G_K$ and $\varphi$. Its residue field $\mathbf{E}' = \mathbf{E}$ or $\tilde{\mathbf{E}}$ is separably closed.

Set $\mathbf{A}'_L = \mathbf{A}'^{G_L}$ ; if $\mathbf{E}'_L = \mathbf{E}'^{G_L}$. Then $\mathbf{A}'_L$ is a complete $p$-adic valuation ring with residue field $\mathbf{E}'_L$.
For any $\mathbb{Z}_p$-adic representation $V$ of $G_K$, define

$$D'_L(V) = (\mathbf{A}' \otimes_{\mathbb{Z}_p} V)^{G_L}$$

and for any $(\varphi, G_\infty)$-module $D$, étale over $\mathbf{A}'_L$,

$$V'_L(D) = (\mathbf{A}' \otimes_{\mathbf{A}'_L} D)^{\varphi=1}.$$

Denote these functors by $D_L$ and $V_L$ when $\mathbf{A}' = \mathbf{A}$ and by $\widetilde{D}_L$ and $\widetilde{V}_L$ when $\mathbf{A}' = \tilde{\mathbf{A}}$. Remark that $D'_L(V)$ and $D(V) \otimes_{\mathbf{A}_K} \mathbf{A}'_L$ are $(\varphi, G_\infty)$-modules over $\mathbf{A}'_L$, the latter being étale. The following theorem shows that they are indeed isomorphic and assures that $D'_L$ is a good equivalent for $D$ in the metabelian case.

**Theorem 1.2.**

1. *The natural map*
$$\iota \; : \; D(V) \otimes_{\mathbf{A}_K} \mathbf{A}'_L \to D'_L(V)$$
*is an isomorphism of $(\varphi, G_\infty)$-modules étale over $\mathbf{A}'_L$.*

2. *Functors $D'_L$ and $V'_L$ are quasi-inverse equivalences of categories between the category of $\mathbb{Z}_p$-adic representations of $G_K$ and the one of étale $(\varphi, G_\infty)$-modules on $\mathbf{A}'_L$.*

*Proof:* First, remark that, because of Theorem 1.1., and after extending scalars, the natural map

$$D(V) \otimes_{\mathbf{A}_K} \mathbf{A}' \to V \otimes_{\mathbb{Z}_p} \mathbf{A}'$$

is an isomorphism.
Taking Galois invariants, we get an isomorphism

$$D(V) \otimes_{\mathbf{A}_K} \mathbf{A}'_L = (D(V) \otimes_{\mathbf{A}_K} \mathbf{A}')^{G_L} \xrightarrow{\sim} (V \otimes_{\mathbb{Z}_p} \mathbf{A}')^{G_L} = D'_L(V)$$

as desired.

We immediately deduce that the functor $D'_L$ from the category of $\mathbb{Z}_p$-adic representations of $G_K$ to the one of étale $(\varphi, G_\infty)$-modules over $\mathbf{A}'_L$ is exact and faithful.
In fact, this result and the expression of the quasi-inverse of $D'_L$ (seen as an equivalence of categories on its essential image) suffice for our use of $(\varphi, G_\infty)$-modules. This quasi-inverse is obtained with the help of the comparison isomorphism after extending scalars:

$$D'_L(V) \otimes_{\mathbf{A}'_L} \mathbf{A}' \simeq D(V) \otimes_{\mathbf{A}_K} \mathbf{A}' \simeq V \otimes_{\mathbb{Z}_p} \mathbf{A}'$$

so that

$$V'_L(D'_L(V)) \simeq V$$

and $V'_L$ is the quasi-inverse of $D'_L$.
Fontaine's computation (cf. [Fon90, Proposition 1.2.6.]) still applies here and permits to compute the essential image which is still the category of étale $(\varphi, G_\infty)$-modules over $\mathbf{A}'_L$. It consists in proving that any $p$-torsion étale $(\varphi, G_\infty)$-module,

which is then an $\mathbf{E}'$-vector space, has a $\varphi$-invariant basis, by showing that for any matrix $(a_{j,l}) \in GL_d(\mathbf{E}')$, the system

$$x_j^p = \sum a_{j,l} x_l$$

admits $p^d$ solutions in $\mathbf{E}'^d$, generating $\mathbf{E}'^d$. The general case is deduced by *dévissage* and passing to the limit. $\square$

**Corollary 1.1.**
*The functor*

$$\begin{array}{ccc} \{\textit{étale}(\varphi, \Gamma_K) - \textit{modules over } \mathbf{A}_K\} & \to & \{\textit{étale}(\varphi, G_\infty) - \textit{modules over } \mathbf{A}'_L\} \\ D & \mapsto & D \otimes_{\mathbf{A}_K} \mathbf{A}'_L \end{array}$$

*is an equivalence of categories.*

**Example** The $(\varphi, G_\infty)$-module associated with the false Tate curve admits a trivial basis $(1 \otimes e_1, 1 \otimes e_2)$. This module is then of finite height over $\mathbf{A}'_L$. It would be interesting to know whether this remains true or not for any semi-stable representation.
When $\mathbf{A}' = \tilde{\mathbf{A}}$, it follows from a result of Kisin ([Kis06, Lemma 2.1.10]). He builds the $\varphi$-module associated with the extension $K_\pi$:

$$(V \otimes_{\mathbb{Z}_p} \mathbf{A}_Y)^{\mathrm{Gal}(\overline{K}/K_\pi)}$$

where $\mathbf{A}_Y$ is the $p$-adic completion of the maximal unramified extension of $W[[Y]][\frac{1}{Y}]$ in $\tilde{\mathbf{A}}$. He shows that semi-stable representations are of finite height in this framework, which means that the $W[[Y]]$-module

$$(V \otimes_{\mathbb{Z}_p} W[[Y]]^{nr})^{\mathrm{Gal}(\overline{K}/K_\pi)},$$

with $W[[Y]]^{nr} = \mathbf{A}_Y \bigcap \tilde{\mathbf{A}}^+$, has the same rank as $V$.

### 1.4.3 Remark: the field of norms of $L/K$

As previously remarked, the extension $L/K$ is arithmetically profinite ; consider then its field of norms $\mathbf{E}_{L/K}$ which can be explicitly described. Indeed if $k_L$ is the residue field of $L$, then there exists $z \in \tilde{\mathbf{E}}$ such that $\mathbf{E}_{L/K}$ identifies with $k_L((z)) \subset \tilde{\mathbf{E}}$. We would then like to reproduce the classical construction of $(\varphi, \Gamma)$-modules by substituting $\mathbf{E}_{L/K}$ to the field of norms of the cyclotomic extension $K_\infty/K$. However, we then have to build a characteristic 0 lift (in $\tilde{\mathbf{A}}$) of $\mathbf{E}_{L/K}$ stable under both actions of $G_K$ and $\varphi$, that we are not able to do. This problem is linked to the fact that we cannot make explicit a norm coherent sequence of uniformizers in the tower $K(\zeta_{p^n}, \pi_{p^n})$.

## 1.5 Galois Cohomology

### 1.5.1 Statement of the theorem

First recall the classical case. Let $D(V)$ be the étale $(\varphi,\ \Gamma_K)$-module over $\mathbf{A}_K$ associated with a $\mathbb{Z}_p$-adic representation $V$. Fix $\gamma$ a topological generator of $\Gamma_K$. Herr introduced in [Her98] the complex

$$0 \longrightarrow D(V) \xrightarrow{f_1} D(V) \oplus D(V) \xrightarrow{f_2} D(V) \longrightarrow 0$$

with maps

$$f_1 = \begin{pmatrix} \varphi - 1 \\ \gamma - 1 \end{pmatrix} \text{ and } f_2 = (\gamma - 1, 1 - \varphi).$$

He showed that the homology of this complex canonically and functorially identifies with the Galois cohomology of the representation $V$.

This identification was explicitly given in [CC99] and [Ben00] for the first cohomology group by associating with the class of a pair $(x, y)$ of elements in $D(V)$ satisfying $(\gamma - 1)x = (\varphi - 1)y$ the class of the cocycle

$$\sigma \mapsto -(\sigma - 1)b + \frac{\gamma^n - 1}{\gamma - 1} y$$

where $b \in V \otimes_{\mathbb{Z}_p} \mathbf{A}$ is a solution of $(\varphi - 1)b = x$ and $\sigma_{|\Gamma_K} = \gamma^n$ for some $n \in \mathbb{Z}_p$.

We will show that there still exists such a complex in the metabelian case. However, in order to take into account that $G_\infty$ has now two generators, we will modify it a little.

Let $M$ be a given étale $(\varphi, G_\infty)$-module over $\mathbf{A}'_L$. Associate with $M$ the four terms complex $C_{\varphi,\gamma,\tau}(M)$:

$$0 \longrightarrow M \xrightarrow{\alpha} M \oplus M \oplus M \xrightarrow{\beta} M \oplus M \oplus M \xrightarrow{\eta} M \longrightarrow 0$$

where

$$\alpha = \begin{pmatrix} \varphi - 1 \\ \gamma - 1 \\ \tau - 1 \end{pmatrix}, \beta = \begin{pmatrix} \gamma - 1 & 1 - \varphi & 0 \\ \tau - 1 & 0 & 1 - \varphi \\ 0 & \tau^{\chi(\gamma)} - 1 & \delta - \gamma \end{pmatrix}, \eta = \begin{pmatrix} \tau^{\chi(\gamma)} - 1, & \delta - \gamma, & \varphi - 1 \end{pmatrix}$$

with $\delta = (\tau^{\chi(\gamma)} - 1)(\tau - 1)^{-1} \in \mathbb{Z}_p[[\tau - 1]]$ defined as follows: set

$$\binom{u}{n} = \frac{u.(u-1)...(u-n+1)}{n!} \in \mathbb{Z}_p \text{ for all } u \in \mathbb{Z}_p \text{ and all } n \in \mathbb{N}.$$

Then:

$$\tau^{\chi(\gamma)} = \sum_{n \geq 0} \binom{\chi(\gamma)}{n} (\tau - 1)^n$$

for $\tau^{p^n}$ converges to 1 in $G_\infty$, and thus $\tau - 1$ is topologically nilpotent in $\mathbb{Z}_p[[G_\infty]]$. So

$$\delta = \frac{\tau^{\chi(\gamma)} - 1}{\tau - 1} = \sum_{n \geq 1} \binom{\chi(\gamma)}{n} (\tau - 1)^{n-1}.$$

The purpose of this paragraph is to show

**Theorem 1.3.**
*Let $V$ be a $\mathbb{Z}_p$-adic representation of $G_K$.*

*i) The homology of the complex $C_{\varphi,\gamma,\tau}(D_L(V))$ canonically and functorially identifies with the continuous Galois cohomology of $V$.*

*ii) Explicitly, let $(x, y, z) \in Z^1(C_{\varphi,\gamma,\tau}(D_L(V)))$, let $b$ be a solution in $V \otimes \mathbf{A}'$ of*

$$(\varphi - 1)b = x,$$

*then the identification above associates with the class of the triple $(x, y, z)$ the class of the cocycle:*

$$c \;:\; \sigma \mapsto c_\sigma = -(\sigma - 1)b + \gamma^n \frac{\tau^m - 1}{\tau - 1} z + \frac{\gamma^n - 1}{\gamma - 1} y$$

*where $\sigma_{|G_\infty} = \gamma^n \tau^m$.*

### 1.5.2 Proof of Theorem 1.3. *i)*

The functor $F^\bullet$ which associates with a $\mathbb{Z}_p$-adic representation $V$ the homology of the complex $C_{\varphi,\gamma,\tau}(D_L(V))$ is a cohomological functor coinciding in degree 0 with the continuous Galois cohomology of $V$:

$$H^0(C_{\varphi,\gamma,\tau}(D_L(V)) = D_L(V)_{\varphi=1,\gamma=1,\tau=1} = V^{G_K}.$$

The proof consists then in showing that it is effaceable. In order to do that, we would like to work with a category with sufficiently many injectives and to see $V$ as a submodule of an explicit injective, its induced module, which is known to be cohomologically trivial. But the category of $\mathbb{Z}_p$-adic representations of $G_K$ doesn't admit induced modules. We will then work modulo $p^r$ for a fixed $r$, and even in the category of direct limits of $p^r$-torsion representations and then deduce the result by passing to the limit. We have then to show that the homology of the complex associated with an induced module concentrates in degree 0, which shows *a fortiori* the effaceability of $F^\bullet$. We will yet write this in an explicit manner, which will let us get the second part of the theorem, and, in the next paragraph, an explicit description of the cup-product in terms of the Herr complex.

Let $M_{G_K,p^r-tor}$ be the category of discrete $p^r$-torsion $G_K$-modules, it is also the category of direct limits of finite $p^r$-torsion $G_K$-modules or also the one of discrete

$\mathbb{Z}/p^r\mathbb{Z}[[G_K]]$-modules. Let us remark that the functor $D_L$ extends to an equivalence of categories from this category to the one of direct limits of $p^r$-torsion étale $(\varphi, G_\infty)$-modules over $\mathbf{A}'_L$.
Note finally that this category is stable under passing to the induced module:

**Lemma 1.1.**
*Let $V$ be an object of $M_{G_K,p^r-tor}$, define the induced module associated with $V$ by:*

$$\mathrm{Ind}_{G_K}(V) := \mathcal{F}_{cont}(G_K, V)$$

*the set of all continuous maps from $G_K$ to $V$.*
*Endow $\mathrm{Ind}_{G_K}(V)$ with the discrete topology and the action of $G_K$:*

$$\begin{array}{rcl} G_K \times \mathrm{Ind}_{G_K}(V) & \to & \mathrm{Ind}_{G_K}(V) \\ g.\eta & = & [x \mapsto \eta(x.g)]. \end{array}$$

*Then $\mathrm{Ind}_{G_K}(V)$ is an object of $M_{G_K,p^r-tor}$ and $V$ canonically injects in $\mathrm{Ind}_{G_K}(V)$.*

*Proof:* The first part of the lemma is well-known. See [TR08] for details. The injection of $V$ in its induced module is given by sending $v \in V$ on $\eta_v \in \mathrm{Ind}_{G_K}(V)$ such that

$$\forall g \in G_K \quad \eta_v(g) = g(v).$$

$\square$

Let $F^i$ denote the composed functor $H^i(C_{\varphi,\gamma,\tau}(D_L(-)))$. The snake lemma gives for any short exact sequence in $M_{G_K,p^r-tor}$

$$0 \to V \to V'' \to V' \to 0$$

a long exact sequence

$$0 \to F^0(V) \to F^0(V'') \to F^0(V') \to F^1(V) \to F^1(V'') \to \cdots$$

which shows that $F^\bullet$ is a cohomological functor.
Let us show that it coincides with the long exact cohomology sequence when $V'' = \mathrm{Ind}_{G_K}(V)$. We use the following result:

**Proposition 1.1.**
*Let $U = \mathrm{Ind}_{G_K}(V)$ be an induced module in the category $M_{G_K,p^r-tor}$, then*

$$F^i(U) = H^i(K, U) = 0 \text{ for all } i > 0.$$

Let us first deduce Point $i)$ of the theorem from this result. The commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & F^0(V) \arrow[r] \arrow[d, equal] & F^0(\mathrm{Ind}_{G_K}(V)) \arrow[r] \arrow[d, equal] & F^0(V') \arrow[r] \arrow[d, equal] & F^1(V) \arrow[r] & 0 \\ 0 \arrow[r] & H^0(K,V) \arrow[r] & H^0(K,\mathrm{Ind}_{G_K}(V)) \arrow[r] & H^0(K,V') \arrow[r] & H^1(K,V) \arrow[r] & 0 \end{tikzcd}$$

shows that $H^1(K,V) \simeq F^1(V)$.
And in higher dimension vanishing of $F^i(\mathrm{Ind}_{G_K}(V))$ and $H^i(K, \mathrm{Ind}_{G_K}(V))$ prove both that $F^k(V') = F^{k+1}(V)$ and $H^k(K,V') = H^{k+1}(K,V)$. Thus, by induction, $F^i(V) = H^i(K,V)$ holds for all $i \in \mathbb{N}$ and for any module $V$ in $M_{G_K,p^r-tor}$.

*Proof of the proposition:*

The Galois cohomology part is a classical result (cf. [Ser68, Ser94] or [TR08]).
For the second part, we will use

**Lemma 1.2.**
*For any $V \in M_{G_K,p^r-tor}$, there is a short exact sequence:*

$$0 \longrightarrow \mathrm{Ind}_{G_\infty}(V) \longrightarrow D_L(\mathrm{Ind}_{G_K}(V)) \xrightarrow{\varphi-1} D_L(\mathrm{Ind}_{G_K}(V)) \longrightarrow 0.$$

*Moreover, for any $\alpha \in \mathbb{Z}_p^*$, there is a short exact sequence:*

$$0 \longrightarrow \mathrm{Ind}_{\Gamma_K}(V) \longrightarrow \mathrm{Ind}_{G_\infty}(V) \xrightarrow{\tau^\alpha-1} \mathrm{Ind}_{G_\infty}(V) \longrightarrow 0.$$

*Finally, there is a short exact sequence:*

$$0 \longrightarrow V^{G_K} \longrightarrow Ind_{\Gamma_K}(V) \xrightarrow{\gamma-1} Ind_{\Gamma_K}(V) \longrightarrow 0.$$

*Proof of the lemma:* Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbf{A}' \xrightarrow{\varphi-1} \mathbf{A}' \longrightarrow 0$$

and tensorize it with $\mathrm{Ind}_{G_K}(V)$. The existence of a continuous section of $\varphi - 1$ (cf. [Sch06]) permits, taking Galois invariants, to get a long exact sequence beginning with

$$0 \longrightarrow \mathrm{Ind}_{G_K}(V)^{G_L} \longrightarrow D_L(\mathrm{Ind}_{G_K}(V)) \xrightarrow{\varphi-1} D_L(\mathrm{Ind}_{G_K}(V)) \longrightarrow H^1(L, \mathrm{Ind}_{G_K}(V))$$

The kernel is given by $\mathrm{Ind}_{G_K}(V)^{G_L} = \mathrm{Ind}_{G_\infty}(V)$.
It remains to show the nullity of $H^1(G_L, \mathrm{Ind}_{G_K}(V))$. Remark (cf.[Ser94, Chapitre I, Proposition 8]):

$$H^1(G_L, \mathrm{Ind}_{G_K}(V)) = \varinjlim H^1(G_M, \mathrm{Ind}_{G_K}(V))$$

where the direct limit is taken over the set of all finite Galois sub-extensions $M$ of $L/K$. Indeed, the sub-Galois groups $G_M$ of $G_K$ form, for inclusion, a projective system with limit

$$\varprojlim G_M = \bigcap G_M = G_L$$

and this system is compatible with the inductive system formed by the $G_M$-modules by restriction $\mathrm{Ind}_{G_K}(V)$ whose limit is the $G_L$-module by restriction $\mathrm{Ind}_{G_K}(V)$.

To prove the lemma, it suffices then to show for any finite Galois extension $M/K$ included in $L$ the vanishing of $H^1(G_M, \mathrm{Ind}_{G_K}(V))$.
But, $G_M$ being open in $G_K$, we have the finite decomposition

$$G_K = \bigcup_{\overline{g} \in \mathrm{Gal}(M/K)} gG_M$$

from which we deduce that, as a $G_M$-module, $\mathrm{Ind}_{G_K}(V)$ admits a decomposition as a direct sum

$$\mathrm{Ind}_{G_K}(V) = \bigoplus_{\overline{g} \in \mathrm{Gal}(M/K)} \mathcal{F}_{cont}(gG_M, V) \simeq \bigoplus_{\mathrm{Gal}(M/K)} \mathrm{Ind}_{G_M}(V).$$

So that

$$H^1(G_M, \mathrm{Ind}_{G_K}(V)) \simeq \bigoplus_{\mathrm{Gal}(M/K)} H^1(G_M, \mathrm{Ind}_{G_M}(V))$$

and any of the $H^1(G_M, \mathrm{Ind}_{G_M}(V))$ is zero, because of the first part of the proposition.

On the other hand, $\tau^\alpha$ topologically generates $\mathrm{Gal}(L/K_\infty)$, so that the complex

$$\mathrm{Ind}_{G_\infty}(V) \xrightarrow{\tau^\alpha - 1} \mathrm{Ind}_{G_\infty}(V)$$

computes the cohomology $H^\bullet(\mathrm{Gal}(L/K_\infty), \mathrm{Ind}_{G_\infty}(V))$. We get the kernel

$$\mathrm{Ind}_{G_\infty}(V)^{\mathrm{Gal}(L/K_\infty)} \simeq \mathrm{Ind}_{\Gamma_K}(V).$$

The vanishing of $H^1(\mathrm{Gal}(L/K_\infty), \mathrm{Ind}_{G_\infty}(V))$ follows from the same arguments as for $H^1(G_L, \mathrm{Ind}_{G_K}(V))$ above.

Finally, the complex

$$\mathrm{Ind}_{\Gamma_K}(V) \xrightarrow{\gamma - 1} \mathrm{Ind}_{\Gamma_K}(V)$$

computes the cohomology $H^\bullet(\Gamma_K, \mathrm{Ind}_{\Gamma_K}(V))$. The surjectivity of $\gamma - 1$ still comes from the nullity of $H^1(\Gamma_K, \mathrm{Ind}_{\Gamma_K}(V))$ which is proved as before. $\Diamond$

From the surjectivity of $(\varphi - 1)$ on $D_L(U)$, we immediately deduce that $F^3(U) = 0$. We also get the kernel of $\eta$:

$$\mathrm{Ker}\ \eta = \{(x, y, z); x, y \in D_L(U) \text{ and } z \in (1-\varphi)^{-1}((\tau^{\chi(\gamma)} - 1)(x) + (\delta - \gamma)(y))\}.$$

Let $x, y \in D_L(U)$ and fix $x', y' \in D_L(U)$ such that

$$(1-\varphi)(x') = x \text{ and } (1-\varphi)(y') = y\ ;$$

proving that $F^2(U) = 0$ consists then in proving

$$\forall u \in \mathrm{Ind}_{G_\infty}(V),\ (x, y, (\tau^{\chi(\gamma)} - 1)(x') + (\delta - \gamma)(y') + u \otimes 1) \in \mathrm{Im}\ \beta.$$

But $(\tau^{\chi(\gamma)}-1)$ is surjective on $\mathrm{Ind}_{G_\infty}(V)$, thus it suffices to consider $\beta(0, x'+u', y')$ with $u'$ chosen so that $(\tau^{\chi(\gamma)}-1)(u') = u$.
Let $(u, v, w) \in \mathrm{Ker}(\beta)$, *i.e.* satisfying:

$$\begin{cases} (\gamma-1)u = (\varphi-1)v \\ (\tau-1)u = (\varphi-1)w \\ (\tau^{\chi(\gamma)}-1)v = (\gamma-\delta)w \end{cases}$$

Fix $x_0 \in D_L(U)$ such that $(\varphi-1)x_0 = u$. Then the first two relations show that

$$v_0 := v - (\gamma-1)x_0 \text{ and } w_0 := w - (\tau-1)x_0$$

lie in the kernel of $\varphi - 1$ thus in $\mathrm{Ind}_{G_\infty}(V)$, and satisfy furthermore:

$$(\tau^{\chi(\gamma)}-1)v_0 = (\gamma-\delta)w_0.$$

Choose now $\eta \in \mathrm{Ind}_{G_\infty}(V)$ such that $(\tau-1)\eta = w_0$. Then

$$(\tau^{\chi(\gamma)}-1)(\gamma-1)\eta = (\gamma-\delta)(\tau-1)\eta = (\tau^{\chi(\gamma)}-1)v_0$$

so that $v_0 - (\gamma-1)\eta \in \mathrm{Ind}_{\Gamma_K}(V)$ and then there exists $\varepsilon \in \mathrm{Ind}_{\Gamma_K}(V)$ such that

$$(\gamma-1)\varepsilon = v_0 - (\gamma-1)\eta$$

so that

$$(\gamma-1)(\eta+\varepsilon) = v_0$$

and

$$(\tau-1)(\eta+\varepsilon) = w_0.$$

Define then $x := x_0 + \eta + \varepsilon$ and let us verify $\alpha(x) = (u, v, w)$:

$$\begin{array}{rclclcl} (\varphi-1)x & = & (\varphi-1)x_0 + (\varphi-1)(\eta+\varepsilon) & = & (\varphi-1)x_0 & = & u \\ (\gamma-1)x & = & (\gamma-1)x_0 + (\gamma-1)(\eta+\varepsilon) & = & v - v_0 + v_0 & = & v \\ (\tau-1)x & = & (\tau-1)x_0 + (\tau-1)(\eta+\varepsilon) & = & w - w_0 + w_0 & = & w \end{array}$$

which proves the proposition. $\square$

### 1.5.3 Explicit Formulas

**Proof of Theorem 1.3.** *ii)*
In order to make the isomorphism explicit, it suffices to do a diagram chasing following the snake lemma: let

$$(x, y, z) \in Z^1(C_{\varphi,\gamma,\tau}(D_L(V))),$$

then through the injection $D_L(V) \hookrightarrow D_L(\mathrm{Ind}_{G_K}(V))$, we can see

$$(x, y, z) \in Z^1(C_{\varphi,\gamma,\tau}(D_L(\mathrm{Ind}_{G_K}(V)))).$$

From the nullity of $H^1(C_{\varphi,\gamma,\tau}(D_L(\mathrm{Ind}_{G_K}(V))))$ we deduce the existence of a $b' \in D_L(\mathrm{Ind}_{G_K}(V))$ such that

$$\alpha(b') = (x, y, z).$$

Consider now $\overline{b'} \in D_L(\mathrm{Ind}_{G_K}(V)/V)$ the reduction of $b'$ modulo $D_L(V)$, then

$$\overline{b'} \in H^0(C_{\varphi,\gamma,\tau}(D_L(\mathrm{Ind}_{G_K}(V)/V))) = (\mathrm{Ind}_{G_K}(V)/V)^{G_K}.$$

Thus, if $\tilde{b} \in \mathrm{Ind}_{G_K}(V)$ lifts $\overline{b'}$, the image of $(x, y, z)$ in $H^1(K, V)$ is the class of the cocycle

$$c : \sigma \mapsto c_\sigma = (\sigma - 1)\tilde{b}.$$

But we can choose $\tilde{b} = b' - b$ since

$$(\varphi - 1)(b' - b) = x - x = 0$$

so that $b' - b \in \mathrm{Ind}_{G_K}(V)$ and then $b' - b$ lifts $\overline{b'}$. So if

$$\sigma_{|G_\infty} = \gamma^n \tau^m,$$

write

$$c_\sigma = (\sigma - 1)(b' - b) = -(\sigma - 1)(b) + (\gamma^n\tau^m - 1)b' = -(\sigma - 1)b + \gamma^n \frac{\tau^m - 1}{\tau - 1} z + \frac{\gamma^n - 1}{\gamma - 1} y$$

which concludes the proof of the theorem.

Let us finally show how to pass to the limit in order to get the result for a representation which is not necessarily torsion. Let $V$ be a $\mathbb{Z}_p$-adic representation of $G_K$. For all $r \geq 1$,

$$V_r = V \otimes \mathbb{Z}/p^r\mathbb{Z}$$

is a $p^r$-torsion representation such that

$$V = \varprojlim V_r.$$

Then we know that the continuous cohomology of $V$ can be expressed as the limit:

$$\forall i \geq 0,\ H^i(K, V) = \varprojlim H^i(K, V_r) = \varprojlim F^i(V_r).$$

It suffices then to show

$$\forall i \geq 0,\ F^i(V) = \varprojlim F^i(V_r).$$

Let $H^i_r$ (respectively $B^i_r$, $Z^i_r$) denote the homology group $H^i(C_{\varphi,\gamma,\tau}(D_L(V_r)))$ (respectively $B^i(C_{\varphi,\gamma,\tau}(D_L(V_r)))$, $Z^i(C_{\varphi,\gamma,\tau}(D_L(V_r))))$. The maps in the Herr complex are $\mathbb{Z}_p$-linear so that in the category of $\mathbb{Z}_p$-modules there is an exact sequence

$$0 \to B^i_r \to Z^i_r \to H^i_r \to 0$$

from which is obtained the exact sequence

$$0 \to \varprojlim B_r^i \to \varprojlim Z_r^i \to \varprojlim H_r^i \to \varprojlim^1 B_r^i$$

where $\varprojlim^1$ is the first derived functor of the functor $\varprojlim$. But for all $r$,

$$B_r^i \simeq B^i(C_{\varphi,\gamma,\tau}(D_L(V))) \otimes \mathbb{Z}/p^r\mathbb{Z}$$

so that the transition maps in the projective system $(B_r^i)$ are surjective, and then this system satisfies Mittag-Leffler conditions. Thus

$$\varprojlim^1 B_r^i = 0$$

shows that the homology of the inverse limit is equal to the inverse limit of the homology, as desired.

**The explicit formula for $H^2$** The isomorphism from $H^2(C_{\varphi,\gamma,\tau}(D_L(V)))$ to $H^2(K,V)$ can as well be made explicit:

**Proposition 1.2.**
*The identification of Theorem 1.3. between the homology of $C_{\varphi,\gamma,\tau}(D_L(V))$ and the Galois cohomology of $V$ associates with a triple $(a,b,c) \in Z^2(C_{\varphi,\gamma,\tau}(D_L(V)))$ the class of the 2-cocycle:*

$$(g,h) \mapsto s_g - s_{gh} + g s_h + \gamma^{n_1}\frac{\tau^{m_1}-1}{\tau-1}\frac{(\delta^{-1}\gamma)^{n_2}-1}{\delta^{-1}\gamma-1}\delta^{-1}c$$

*where $g_{|G_\infty} = \gamma^{n_1}\tau^{m_1}$, $h_{|G_\infty} = \gamma^{n_2}\tau^{m_2}$ and $s$ is a map $G_K \to \mathbf{A}' \otimes V$ such that*

$$s_\sigma = \phi\left(\frac{\gamma^n-1}{\gamma-1}a + \gamma^n\frac{\tau^m-1}{\tau-1}b\right)$$

*where $\sigma_{|G_\infty} = \gamma^n\tau^m$ and $\phi$ is a continuous section of $\varphi-1$.*

*Proof:* The proof is, mutatis mutandis, the same as the above one and can be found in [TR08].

**Remark**
In the classical Herr complex case, with the class of $a$ is associated the class of the 2-cocycle:

$$(g_1,g_2) \mapsto \tilde{\gamma}^{n_1}(h-1)\frac{\tilde{\gamma}^{n_2}-1}{\tilde{\gamma}-1}\tilde{a}$$

where $(\varphi-1)\tilde{a} = a$, $\tilde{\gamma}$ is a fixed lift of $\gamma$ in $G_K$ and $g_1 = \tilde{\gamma}^{n_1}h$, $g_2 = \tilde{\gamma}^{n_2}h'$ with $h,h' \in G_{K_\infty}$ and $n_1,n_2 \in \mathbb{Z}_p$.

## 1.6 The cup-product

### 1.6.1 Explicit formulas for the cup-product

In [Her01], Herr gave explicit formulas for the cup-product in terms of the complex associated with the representation. The following theorem gives the formulas obtained in the metabelian case:

**Theorem 1.4.**
*Let $V$ and $V'$ be two $\mathbb{Z}_p$-adic representations of $G_K$, then the cup-product induces maps:*

1. *Let $(a) \in H^0(C_{\varphi,\gamma,\tau}(D_L(V)))$ and $(a') \in H^0(C_{\varphi,\gamma,\tau}(D_L(V')))$,*
$$(a) \cup (a') = (a \otimes a') \in H^0(C_{\varphi,\gamma,\tau}(D_L(V \otimes V'))),$$
2. *let $(x, y, z) \in H^1(C_{\varphi,\gamma,\tau}(D_L(V)))$ and $(a') \in H^0(C_{\varphi,\gamma,\tau}(D_L(V')))$,*
$$(x, y, z) \cup (a') = (x \otimes a', y \otimes a', z \otimes a') \in H^1(C_{\varphi,\gamma,\tau}(D_L(V \otimes V'))),$$
3. *let $(a) \in H^0(C_{\varphi,\gamma,\tau}(D_L(V)))$ and $(x', y', z') \in H^1(C_{\varphi,\gamma,\tau}(D_L(V')))$,*
$$(a) \cup (x', y', z') = (a \otimes x', a \otimes y', a \otimes z') \in H^1(C_{\varphi,\gamma,\tau}(D_L(V \otimes V')))$$
4. *let $(x, y, z) \in H^1(C_{\varphi,\gamma,\tau}(D_L(V)))$ and $(x', y', z') \in H^1(C_{\varphi,\gamma,\tau}(D_L(V')))$, $(x, y, z) \cup (x', y', z') \in H^2(C_{\varphi,\gamma,\tau}(D_L(V \otimes V')))$ can be written as:*
$$(y \otimes \gamma x' - x \otimes \varphi y' \ , \ z \otimes \tau x' - x \otimes \varphi z' \ , \ \delta z \otimes \tau^{\chi(\gamma)} y' - y \otimes \gamma z' + \Sigma_{z,z'})$$
*where*
$$\Sigma_{z,z'} = \sum_{n \geq 1} \binom{\chi(\gamma)}{n+1} \sum_{k=1}^{n} \binom{n}{k} (\tau - 1)^{k-1} z \otimes \tau^k (\tau - 1)^{n-k} z').$$

### 1.6.2 Proof of Theorem 1.4.

The only non trivial identity is the last one. We will use the construction of the previous paragraph and we can then suppose that $V$ and $V'$ are objects of $M_{G_K, p^r - tor}$. We will use the exact sequences

$$0 \to V \to \mathrm{Ind}_{G_K}(V) \to V'' \to 0$$

and

$$0 \to F^0(V) \to F^0(\mathrm{Ind}_{G_K}(V)) \to F^0(V'') \to F^1(V) \to 0$$

and the cup-product property $da \cup b = d(a \cup b)$.
More precisely, fix $(x, y, z)$ and $(x', y', z')$ as in the theorem. Then there exists an

element $a \in D_L(\mathrm{Ind}_{G_K}(V))$ satisfying $\alpha(a) = (x, y, z)$ and $\overline{a} \in (\mathrm{Ind}_{G_K}(V)/V)^{G_K}$. Then $(x, y, z) \cup (x', y', z')$ is equal to

$$\begin{aligned}
\alpha(a) \cup (x', y', z') &= d(\overline{a} \otimes x', \overline{a} \otimes y', \overline{a} \otimes z') \;=\; \beta(a \otimes x', a \otimes y', a \otimes z') \\
&= ((\gamma - 1)(a \otimes x') - (\varphi - 1)(a \otimes y'), \\
&\quad\;(\tau - 1)(a \otimes x') - (\varphi - 1)(a \otimes z'), \\
&\quad\;(\tau^{\chi(\gamma)} - 1)(a \otimes y') - (\gamma - \delta)(a \otimes z'))
\end{aligned}$$

Now we use the formal identity

$$(\sigma - 1)(a \otimes b) = (\sigma - 1)a \otimes \sigma b + a \otimes (\sigma - 1)b.$$

The first term can be written as

$$\begin{aligned}
(\gamma - 1)a \otimes x' \;&-\; (\varphi - 1)a \otimes y' \\
&= (\gamma - 1)a \otimes \gamma x' + a \otimes (\gamma - 1)x' - (\varphi - 1)a \otimes y' - a \otimes (\varphi - 1)y' \\
&= y \otimes \gamma x' + a \otimes ((\gamma - 1)x' - (\varphi - 1)y') - x \otimes y' \\
&= y \otimes \gamma x' - x \otimes y'.
\end{aligned}$$

From a similar computation, we get for the second one

$$\begin{aligned}
(\tau - 1)(a \otimes x') \;&-\; (\varphi - 1)(a \otimes z') \\
&= (\tau - 1)a \otimes \tau x' + a \otimes (\tau - 1)x' - (\varphi - 1)a \otimes z' - a \otimes (\varphi - 1)z' \\
&= z \otimes \tau x' + a \otimes ((\tau - 1)x' - (\varphi - 1)z') - x \otimes z' \\
&= z \otimes \gamma x' - x \otimes z'.
\end{aligned}$$

Let us finally write the computation of the third term.
Iterating the identity

$$(\sigma - 1)(a \otimes b) = (\sigma - 1)a \otimes \sigma b + a \otimes (\sigma - 1)b,$$

we get by induction:

$$(\sigma - 1)^n (a \otimes b) = \sum_{k=0}^{n} \binom{n}{k} (\sigma - 1)^k a \otimes \sigma^k (\sigma - 1)^{n-k} b.$$

First:

$$(\tau^{\chi(\gamma)} - 1)a \otimes y' = (\tau^{\chi(\gamma)} - 1)a \otimes \tau^{\chi(\gamma)} y' + a \otimes (\tau^{\chi(\gamma)} - 1)y' = \delta z \otimes \tau^{\chi(\gamma)} y' + a \otimes (\gamma - \delta) z'$$

and

$$(\gamma - 1)a \otimes z' = (\gamma - 1)a \otimes \gamma z' + a \otimes (\gamma - 1)z' = y \otimes \gamma z' + a \otimes (\gamma - 1)z'.$$

It remains to compute $\delta(a \otimes z')$. Recall that

$$\delta = \frac{\tau^{\chi(\gamma)} - 1}{\tau - 1} = \sum_{n \geq 1} \binom{\chi(\gamma)}{n} (\tau - 1)^{n-1}.$$

So

$$\begin{aligned}
\delta(a \otimes z') &= \sum_{n \geq 1} \binom{\chi(\gamma)}{n} (\tau - 1)^{n-1} (a \otimes z') \\
&= \sum_{n \geq 1} \binom{\chi(\gamma)}{n} \sum_{k=0}^{n-1} \binom{n-1}{k} (\tau - 1)^k a \otimes \tau^k (\tau - 1)^{n-1-k} z' \\
&= a \otimes \delta z' + \sum_{n \geq 1} \binom{\chi(\gamma)}{n} \sum_{k=1}^{n-1} \binom{n-1}{k} (\tau - 1)^{k-1} z \otimes \tau^k (\tau - 1)^{n-1-k} z'.
\end{aligned}$$

Which gives the result. $\square$

## 1.7 Kummer's map

In this paragraph, we suppose $p$ is odd and $\mathbf{A}' = \tilde{\mathbf{A}}$.
The purpose is to compute, in terms of the Herr complex, Kummer's map

$$\kappa : K^* \to H^1(K, \mathbb{Z}_p(1)).$$

More precisely, let

$$F(Y) \in \left( W[[Y]][\frac{1}{Y}] \right)^{\times},$$

we will compute a triple $(x, y, z) \in Z^1(C_{\varphi,\gamma,\tau}(\tilde{\mathbf{A}}_L(1)))$ corresponding to the image $\kappa o \theta(F(Y))$ of

$$\theta(F(Y)) = F(\pi) \in K^*.$$

Remark that there exist $d \in \mathbb{Z}$ and $G(Y) \in (W[[Y]])^{\times}$ such that

$$F(Y) = Y^d G(Y).$$

In fact $G(Y)$ can be written as the product of a $p$th root of unity (which doesn't play any role) and a series in $1 + (p) \subset W[[Y]]$.
Denote

$$\alpha = \theta(F(Y)) \in K^*.$$

Choose

$$\tilde{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots) \in \tilde{\mathbf{E}}$$

such that $\alpha_0 = \alpha$. Then

$$\frac{\tilde{\alpha}}{\rho^d} \in \tilde{\mathbf{E}}^+$$

thus
$$\frac{[\tilde{\alpha}]}{Y^d} \in \tilde{\mathbf{A}}^+$$
and for all $\sigma \in G_K$, there exists $\psi_\alpha(\sigma) \in \mathbb{Z}_p$ such that
$$\sigma(\alpha) = \alpha\varepsilon^{\psi_\alpha(\sigma)}.$$
The map $\sigma \mapsto \varepsilon^{\psi_\alpha(\sigma)}$ is in fact a cocycle computing $\kappa(\alpha)$. So
$$\sigma([\tilde{\alpha}]) = [\tilde{\alpha}](1+X)^{\psi_\alpha(\sigma)} \text{ where } \kappa(\alpha) = \varepsilon^{\psi_\alpha} \in H^1(K, \mathbb{Z}_p(1)).$$
On the other hand, the series $\log \frac{[\tilde{\alpha}]}{F(Y)}$ converges in $B_{crys}$ and even in $\mathrm{Fil}^1 B_{crys}$, namely $\frac{[\tilde{\alpha}]}{F(Y)} \in \tilde{\mathbf{A}}^+$ and $\theta\left(\frac{[\tilde{\alpha}]}{F(Y)}\right) = 1$.
For all $h \in G_L$,
$$(h-1)\log\frac{[\tilde{\alpha}]}{F(Y)} = \psi_\alpha(h)t \text{ where } t = \log(1+X).$$
Define
$$\tilde{b} = \log\frac{[\tilde{\alpha}]}{F(Y)}/t \in \mathrm{Fil}^0 B_{crys}.$$
Then
$$\psi_\alpha(h) = (h-1)(\tilde{b}) \ \forall h \in G_L.$$
Let
$$f(Y) = \mathscr{L}(F) = \frac{1}{p}\log\frac{F(Y)^p}{\varphi(F(Y))} \in W[[Y]]$$
then
$$(\varphi-1)(\tilde{b}) = \frac{1}{t}f(Y).$$
Choose $b_1 \in \tilde{\mathbf{A}}$ a solution of
$$(\varphi-1)b_1 = -\frac{f(Y)}{X}.$$
Let $X_1 = \varphi^{-1}(X) = [\varepsilon^{\frac{1}{p}}] - 1$, and $\omega = \frac{X}{X_1} \in \tilde{\mathbf{A}}^+$ then
$$(\varphi-\omega)(b_1X_1) = -f(Y).$$
But reducing modulo $p$ this identity yields to an equation of the form
$$T^p - \overline{\omega}T = -\overline{f(Y)}$$
and then by successive approximations modulo $p^m$, and because $\tilde{\mathbf{E}}^+$ is integrally closed, $b_1X_1 \in \tilde{\mathbf{A}}^+$. But $\frac{1}{X_1} \in \mathrm{Fil}^0 B_{crys}$, namely the series
$$\frac{t}{X_1} = \sum_{n>0}(-1)^{n+1}\frac{\omega X^{n-1}}{n} = \sum_{n>0}(-1)^{n+1}\frac{\omega^n X_1^{n-1}}{n}$$

converges in $\mathrm{Fil}^1 A_{crys}$, and thus

$$\frac{1}{X_1} = \frac{t}{X_1}\frac{1}{t} \in \mathrm{Fil}^0 B_{crys}.$$

So

$$b_1 = (b_1 X_1).\frac{1}{X_1} \in \mathrm{Fil}^0 B_{crys}.$$

Moreover, $(\varphi - 1)b_2 = -\frac{f(Y)}{2}$ admits a solution $b_2$ in $\tilde{\mathbf{A}}^+$, so that if we set

$$x = -\frac{f(Y)}{X} - \frac{f(Y)}{2} \in \tilde{\mathbf{A}}_L$$

and choose a solution $b \in \tilde{\mathbf{A}}$ of $(\varphi - 1)b = x$, then $b \in \mathrm{Fil}^0 B_{crys}$.
So $\tilde{b} + b \in \mathrm{Fil}^0 B_{crys}$ and

$$(\varphi - 1)(\tilde{b} + b) = (\frac{1}{t} - \frac{1}{X} - \frac{1}{2})f(Y).$$

And we have the following lemma:

**Lemma 1.3.**
*Solutions of the equation*

$$(\varphi - 1)(\mu) = \left(\frac{1}{t} - \frac{1}{X} - \frac{1}{2}\right) f(Y) \tag{1.1}$$

*in* $\mathrm{Fil}^0 B_{crys}$ *lie in* $\mathbb{Q}_p + \mathrm{Fil}^1 B_{crys}$ *and are invariant under the action of* $G_L$.

*Proof of the lemma:* Consider

$$\begin{aligned} u &= t\left(\frac{1}{t} - \frac{1}{X} - \frac{1}{2}\right) f(Y) = \left(1 - \frac{t}{X} - \frac{t}{2}\right) f(Y) \\ &= -\sum_{n\geq 2} \frac{(-X)^n}{n+1} f(Y) + \sum_{n\geq 2} \frac{(-X)^n}{2n} f(Y) \end{aligned}$$

then letting $\mu' = t\mu$, Equation (1.1) becomes

$$\left(\frac{\varphi}{p} - 1\right)(\mu') = u \tag{1.2}$$

but the sequences $\frac{(-X)^n}{n+1} f(Y)$ and $\frac{(-X)^n}{2n} f(Y)$ converge to 0 in $B_{crys}$ and

$$\left(\frac{\varphi}{p}\right)^k \left(\frac{X^n}{n+1}\right) = \frac{((1+X)^{p^k} - 1)^n}{(n+1)p^k}$$

but

$$((1+X)^{p^k} - 1) = \sum_{1\leq r\leq p^k} \frac{p^k!}{(p^k - r)!}\frac{X^r}{r!} \in p^k A_{crys}$$

so

$$\left(\frac{\varphi}{p}\right)^k\left(\frac{X^n}{n+1}\right)\in\frac{p^{k(n-1)}}{n+1}A_{crys}$$

converges to 0 uniformly in $n$ in $B_{crys}$. The same holds for $\left(\frac{\varphi}{p}\right)^k\left(\frac{X^n}{n+1}\right)$. So we get a solution $-\sum_{n\geq 0}\left(\frac{\varphi}{p}\right)^n u$ of Equation (1.2) in $(\mathrm{Fil}^2 B_{crys})^{G_L}$ thus a solution of Equation (1.1) in $(\mathrm{Fil}^1 B_{crys})^{G_L}$. And the fact that

$$(\mathrm{Fil}^0 B_{crys})_{\varphi=1}=\mathbb{Q}_p$$

proves the lemma. $\Diamond$

So $b+\tilde{b}\in(\mathrm{Fil}^0 B_{crys})^{G_L}$, thus, for all $h\in G_L$,

$$(h-1)(-b)=(h-1)\tilde{b}=\psi_\alpha(h).$$

We conclude that there exist a unique $z\in\tilde{\mathbf{A}}_L(1)$ and $y\in\tilde{\mathbf{A}}_L(1)$ unique modulo $(\gamma-1)\mathbb{Z}_p(1)$ such that $\kappa(\alpha)$ is the image in $H^1(K,\mathbb{Z}_p(1))$ of the triple

$$(x,y,z)\in Z^1(C_{\varphi,\gamma,\tau}(\tilde{\mathbf{A}}_L(1)))$$

where $x=-(\frac{1}{X}+\frac{1}{2})f(Y)\otimes\varepsilon$. Namely, we know that there exists such a triple $(x',y',z')$, and

$$x'-x\in(\varphi-1)\tilde{\mathbf{A}}_L(1)$$

which shows the existence, and $x$ being fixed, the unicity modulo $\alpha(\mathbb{Z}_p)$ (where $\alpha$ is the first map in the Herr complex $C_{\varphi,\gamma,\tau}(M)$, cf. section 1.5).

We get the more precise result:

**Proposition 1.3.**
*Let $F(Y)\in\left(W[[Y]][\frac{1}{Y}]\right)^\times$. Then the image of $F(\pi)$ by Kummer's map corresponds to the class of a triple*

$$\left(-f(Y)\left(\frac{1}{X}+\frac{1}{2}\right),y,z\right)\otimes\varepsilon$$

*with $y,z\in W[[X,Y]]$. This triple is congruent modulo $XYW[[X,Y]]$ to*

$$\left(-\frac{f(Y)}{X}-\frac{f(Y)}{2},\ 0,\ Y d_{\log}F(Y)\right)\otimes\varepsilon$$

*where $d_{\log}$ stands for the logarithmic derivative.*

*Proof:* We have to show the congruences.

Remark that

$$\begin{aligned}\gamma\left(\frac{1\otimes\varepsilon}{X}\right) &= \frac{\chi(\gamma)\otimes\varepsilon}{\chi(\gamma)X+\frac{\chi(\gamma)(\chi(\gamma)-1)}{2}X^2+X^3u(X)}\\ &= \left(\frac{1}{X}-\frac{(\chi(\gamma)-1)}{2}+Xv(X)\right)\otimes\varepsilon\end{aligned}$$

so that

$$(\gamma - 1)x \in XYW[[X, Y]](1)$$

where $\varphi^n$ is topologically nilpotent thus $\varphi - 1$ is invertible. Then it comes

$$y \in \mathbb{Z}_p(1) + XYW[[X, Y]](1).$$

Moreover, let $\tilde{\gamma}$ lift $\gamma$ in $G_K$, we still have

$$(\tilde{\gamma} - 1)(\tilde{b} \otimes \varepsilon) = \psi_\alpha(\tilde{\gamma})$$

where, because of $ii)$ of Theorem 1.3. on the one hand, and Lemma 1.3. above on the other hand,

$$(\tilde{\gamma} - 1)(\tilde{b} \otimes \varepsilon + b \otimes \varepsilon) = \psi_\alpha(\tilde{\gamma}) + (\tilde{\gamma} - 1)(b \otimes \varepsilon) = y \in \mathrm{Fil}^1 B_{crys}(1)$$

which shows that

$$y \in XYW[[X, Y]](1).$$

We proceed as well for $z$:

$$(\tau - 1)f(Y) = (f(Y(1+X)) - f(Y)) = \sum_{n \geq 1} \frac{(XY)^n}{n!} f^{(n)}(Y) \equiv XYf'(Y) \mod (XY)^2.$$

Remark moreover

$$\left(Y \frac{d}{dY}\right) \circ \frac{\varphi}{p} = \varphi \circ \left(Y \frac{d}{dY}\right)$$

so that

$$(\tau - 1)f(Y) \equiv X(1 - \varphi)\left(Y d_{\log} F(Y)\right) \mod (XY)^2$$

and thus

$$(\tau - 1)x \equiv (\varphi - 1)(Y d_{\log} F(Y) \otimes \varepsilon) \mod XYW[[X, Y]](1)$$

which shows

$$z \in Y d_{\log} F(Y) \otimes \varepsilon + \mathbb{Z}_p(1) + XYW[[X, Y]](1). \tag{1.3}$$

And if $\tilde{\tau}$ lifts $\tau$ in $G_K$,

$$(\tilde{\tau} - 1)(\tilde{b} + b) = \psi_\alpha(\tilde{\tau}) - \log \frac{F(Y(1+X))}{F(Y)}/t + (\tilde{\tau} - 1)b \in \mathrm{Fil}^1 B_{crys}$$

so that

$$z = \psi_\alpha(\tilde{\tau}) + (\tilde{\tau} - 1)b \in \log \frac{F(Y(1+X))}{F(Y)}/t + \mathrm{Fil}^1 B_{crys}$$

which, combined with (1.3), proves the desired result. □

# 2 Formal Groups

In this section, we will prove the Brückner-Vostokov explicit formula for formal groups. In [Abr97], Abrashkin proved this formula under the condition that the $p^M$-th roots of unity belong to the base field, which turns out not to be necessary. To prove this formula without this assumption, we will explicitly compute the Kummer map linked to the Hilbert symbol of a formal group in terms of its $(\varphi, \Gamma)$-module, then compute the cup-product with the usual Kummer map and the image of this cup-product through the reciprocity isomorphism, which gives the desired formula.

## 2.1 Notation and backgrounds on formal groups

Consider $G$ a connected smooth formal group over $W = W(k)$, the ring of Witt vectors with coefficients in the finite field $k$. Denote by $K_0$ the fraction field of $W$ and $K$ a totally ramified extension of $K_0$. Under these hypotheses, one can associate (cf. [Fon77]) with $G$ a formal group law which determines $G$. Let us recall what it is.

### 2.1.1 Formal group laws

Fix $p$ an odd prime and $d > 0$ a number. Write $\mathbf{X} = (X_1, \ldots, X_d)$, $\mathbf{Y} = (Y_1, \ldots, Y_d)$ and $\mathbf{Z} = (Z_1, \ldots, Z_d)$.

**Definition 2.1.**
*A (commutative)* formal group law $\mathbf{F}$ *of dimension $d$ on a commutative ring $R$ is the data of a $d$-uple of formal power series*

$$\mathbf{F}(\mathbf{X}, \mathbf{Y}) = (F_i(X_1, \ldots, X_d, Y_1, \ldots, Y_d))_{1 \leq i \leq d} \in (R[[\mathbf{X}, \mathbf{Y}]])^d$$

*satisfying*

1. $\mathbf{F}(\mathbf{X}, \mathbf{0}) = \mathbf{F}(\mathbf{0}, \mathbf{X}) = \mathbf{X}$,
2. $\mathbf{F}(\mathbf{X}, \mathbf{F}(\mathbf{Y}, \mathbf{Z})) = \mathbf{F}(\mathbf{F}(\mathbf{X}, \mathbf{Y}), \mathbf{Z})$,
3. $\mathbf{F}(\mathbf{X}, \mathbf{Y}) = \mathbf{F}(\mathbf{Y}, \mathbf{X})$.

For a given formal group law $\mathbf{F}$, there exists a $d$-uple $\mathbf{f} \in (R[[\mathbf{X}]])^d$ such that

$$\mathbf{F}(\mathbf{X}, \mathbf{f}(\mathbf{X})) = \mathbf{F}(\mathbf{f}(\mathbf{X}), \mathbf{X}) = \mathbf{0}$$

so that on a given area where $\mathbf{F}$ and $\mathbf{f}$ converge (for instance $\mathfrak{m}_R^d$, when $R$ is a local ring with maximal ideal $\mathfrak{m}_R$, complete for the $\mathfrak{m}_R$-adict opology), $\mathbf{F}$ defines a commutative group structure, with identity element $\mathbf{0}$, the inverse of $\mathbf{x}$ being $\mathbf{f}(\mathbf{x})$. We then denote the group law by

$$\mathbf{x} +_F \mathbf{y} := \mathbf{F}(\mathbf{x}, \mathbf{y}).$$

Let $\mathbf{G}$ be another formal group law over $R$ of dimension $d'$. Then a morphism from $\mathbf{F}$ to $\mathbf{G}$ is a $d'$-uple of formal series $\mathbf{h}(\mathbf{X}) \in (R[[\mathbf{X}]])^{d'}$ with no constant term such that

$$\mathbf{h}(\mathbf{F}(\mathbf{X}, \mathbf{Y})) = \mathbf{G}(\mathbf{h}(\mathbf{X}), \mathbf{h}(\mathbf{Y})).$$

A morphism $\mathbf{h}$ is an isomorphism if $d = d'$ and if there exists $\mathbf{g}(\mathbf{X}) \in (R[[\mathbf{X}]])^d$ with no constant term satisfying

$$\mathbf{h} \circ \mathbf{g}(\mathbf{X}) = \mathbf{g} \circ \mathbf{h}(\mathbf{X}) = \mathbf{X}$$

or equivalently if $d\mathbf{h}(\mathbf{0}) \in GL_d(R)$. Call $\mathbf{h}$ a strict isomorphism when the normalization $d\mathbf{h}(\mathbf{0}) = I_d$ holds, *i.e.*, if for all $1 \leq i \leq d$, $h_i(\mathbf{X}) \equiv X_i \mod \deg 2$.
When $R$ is an algebra over $\mathbb{Q}$, then any formal group $\mathbf{F}$ admits a unique strict isomorphism, denoted by $\log_F$ from $\mathbf{F}$ to the additive formal group law $\mathbf{F}_a : (\mathbf{X}, \mathbf{Y}) \mapsto \mathbf{X} + \mathbf{Y}$. Call this isomorphism the *vectorial logarithm* of $\mathbf{F}$.
Coordinate maps of $\log_F$ form a basis of the *logarithms* of $\mathbf{F}$, the morphisms from $\mathbf{F}$ to the additive group on $R$.

### 2.1.2 $p$-adic periods

Let us recall the notation of the first part: $K$ is a finite extension of $\mathbb{Q}_p$ with residue field $k$ and $K_0 = W(k)[\frac{1}{p}]$. Fix $M \in \mathbb{N}$.

**Definition 2.2.**
*If the isogeny* $p\mathrm{id}_G : G \to G$ *is finite and flat over* $W$ *of degree* $p^h$*, then* $G$ *is said to be* of finite height *and* $h$ *is called the* height *of* $G$*.*

Let $G$ be a formal group over $W$ of dimension $d$ and finite height $h$. Define

$$G[p^n] = \ker(p^n \mathrm{id}_G : G \to G)$$

the sub-formal group of $p^n$-torsion points of $G$ and denote

$$T(G) = \lim_{\leftarrow} G[p^n](\overline{K})$$

the Tate module of $G$. Suppose moreover that

$$G[p^M](\overline{K}) = G[p^M](K),$$

that is, suppose $p^M$-torsion points of $G$ lie in $K$.
Then $T(G)$ is a free $\mathbb{Z}_p$-module of rank $h$ and $G[p^M](\overline{K}) = G[p^M](K)$ is isomorphic as a $\mathbb{Z}_p$-adic representation of $G_K$ to $(\mathbb{Z}/p^M\mathbb{Z})^h$.
The space of pseudo-logarithms of $G$ (on $K_0$) is the quotient

$$\{F \in K_0[[\mathbf{X}]] \mid F(\mathbf{X} +_G \mathbf{Y}) - F(\mathbf{X}) - F(\mathbf{Y}) \in \mathcal{O}_{K_0}[[\mathbf{X}, \mathbf{Y}]] \otimes \mathbb{Q}_p\} \ /\mathcal{O}_{K_0}[[\mathbf{X}]] \otimes \mathbb{Q}_p.$$

Denote it by $H^1(G)$. It is a $K_0$-vector space of dimension $h$. The space of logarithms of $G$ is

$$\Omega(G) = \{F \in K_0[[\mathbf{X}]] \mid F(\mathbf{X} +_G \mathbf{Y}) = F(\mathbf{X}) + F(\mathbf{Y})\}.$$

It is naturally a sub-$K_0$-vector space of $H^1(G)$ of dimension $d$. Moreover, $H^1(G)$ admits the filtration

$$\mathrm{Fil}^0(H^1(G)) = H^1(G), \quad \mathrm{Fil}^1(H^1(G)) = \Omega(G), \quad \mathrm{Fil}^2(H^1(G)) = 0.$$

With its filtration, and the Frobenius:

$$\varphi : F(\mathbf{X}) \mapsto F^{\varphi}(\mathbf{X}^p),$$

$H^1(G)$ is called the *Dieudonné module* of $G$.
In [Fon77], Fontaine showed there exists a pairing

$$H^1(G) \times T(G) \to B^+_{crys}$$

explicitly described by Colmez in [Col92].
It is defined as follows: let $\overline{F} \in H^1(G)$, and $o = (o_s)_{s\geq 0} \in T(G)$ ; choose for all $s$ a lift $\hat{o}_s \in W(\mathfrak{m}_{\tilde{\mathbf{E}}})^d$ of $o_s$, *i.e.* satisfying $\theta(\hat{o}_s) = o_s$. Then the sequence $p^s F(\hat{o}_s)$ converges to an element $\int_o d\overline{F}$ in $B^+_{crys}$ independent of the choice of lifts $\hat{o}_s$ and $F$. Moreover, this pairing is compatible with actions of Galois and $\varphi$ and with filtrations: if $F$ is a logarithm, then $\int_o dF \in \mathrm{Fil}^1 B^+_{crys}$.
This pairing permits to identify $H^1(G)$ with $\mathrm{Hom}_{G_{K_0}}(T(G), B^+_{crys})$ with the filtration induced by the one of $B^+_{crys}$. In order to work at an entire level, let us introduce a lattice of $H^1(G)$, the $W$-module

$$D^*_{crys}(G) = \mathrm{Hom}_{G_{K_0}}(T(G), A_{crys})$$

endowed with the filtration and the Frobenius $\varphi$ induced by those on $A_{crys}$. The functor $D^*_{crys}$ is a contravariant version of the crystalline functor of Fontaine's theory. The filtration is of length 1 and we denote

$$D^0(G) = D^*_{crys}(G) = \mathrm{Hom}_{G_{K_0}}(T(G), A_{crys})$$

and

$$D^1(G) = \mathrm{Fil}^1 D^*_{crys}(G) = \mathrm{Hom}_{G_{K_0}}(T(G), \mathrm{Fil}^1 A_{crys}).$$

Then $D^1(G)$ is a direct factor of $D^0(G)$ of rank $d$. Fix then a basis $\{l_1, \ldots, l_d\}$ of $D^1(G)$ completed into a basis

$$\{l_1, \ldots, l_d, m_1, \ldots, m_{h-d}\}$$

of $D^0(G)$.
For all $1 \leq i \leq d$, $\varphi(l_i)$ takes values in

$$\varphi(\mathrm{Fil}^1 A_{crys})^d \subset (pA_{crys})^d$$

so, $\frac{\varphi}{p}(l_i)$ takes values in $D^0(G)$. Moreover, [Fon77] and [FL82] show on the one hand that $\varphi$ is topologically nilpotent on $D^0(G)$ (because $G$ is connected) and on the other hand that the filtered module $D^0(G)$ satisfies

$$D^0(G) = \varphi D^0(G) + \frac{\varphi}{p} D^1(G).$$

So, define $\tilde{\varphi}$ the endomorphism of $D^0$ by

$$\begin{array}{rcl} \tilde{\varphi}(l_i) & = & \frac{\varphi}{p}(l_i) \ \forall\ 1 \leq i \leq d, \qquad \text{and} \\ \tilde{\varphi}(m_i) & = & \varphi(m_i) \ \forall\ 1 \leq i \leq h-d, \end{array}$$

then its matrix $\mathcal{E} \in GL_h(W)$.
Let $\mathbf{l} = {}^t(l_1, \ldots, l_n)$ and $\mathbf{m} = {}^t(m_1, \ldots, m_{h-n})$, then

$$\begin{pmatrix} \frac{\varphi}{p}(\mathbf{l}) \\ \varphi(\mathbf{m}) \end{pmatrix} = \mathcal{E} \begin{pmatrix} \mathbf{l} \\ \mathbf{m} \end{pmatrix}.$$

So, we can write a block decomposition

$$\mathcal{E}^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

so that

$$\mathbf{l} = A\frac{\varphi}{p}(\mathbf{l}) + B\varphi(\mathbf{m}) \text{ and } \mathbf{m} = C\frac{\varphi}{p}(\mathbf{l}) + D\varphi(\mathbf{m}).$$

But $\varphi$ is topologically nilpotent on $D^0(G)$, and we can write

$$\mathbf{l} = \sum_{u \geq 1} F_u \frac{\varphi^u(\mathbf{l})}{p}, \quad \mathbf{m} = \sum_{u \geq 1} F'_u \frac{\varphi^u(\mathbf{l})}{p} \tag{2.4}$$

where

$$F_1 = A,\ F_2 = B\varphi(C),\ F_u = B\left(\prod_{1 \leq k \leq u-2} \varphi^k(D)\right)\varphi^{u-1}(C) \text{ for } u > 2,$$

and

$$F'_1 = C,\ F'_2 = D\varphi(C),\ F'_u = (\prod_{0 \leq k \leq u-2} \varphi^k(D))\varphi^{u-1}(C).$$

Define a $\mathbb{Z}_p$-linear operator

$$\mathcal{A} = \sum_{u \geq 1} F_u \varphi^u$$

on $K_0[[\mathbf{X}]]^d$. The vectorial formal power series

$$l_{\mathcal{A}}(\mathbf{X}) = \mathbf{X} + \sum_{m \geq 1} \frac{\mathcal{A}^m(\mathbf{X})}{p^m}$$

gives then the vectorial logarithm of a formal group $F$ from which we can recover the formal group law $\mathbf{F}$ by:

$$\mathbf{F}(\mathbf{X},\mathbf{Y}) = l_{\mathcal{A}}^{-1}\left(l_{\mathcal{A}}(\mathbf{X}) + l_{\mathcal{A}}(\mathbf{Y})\right).$$

In [Hon70], Honda introduced the *type* of a logarithm. A logarithm log is of type $u \in M_d(W)[[\varphi]]$ if $u$ is special, i.e. $u \equiv pI_d \mod \varphi$ and if

$$u(\log) \equiv 0 \mod p.$$

We remark that $pI_d - \mathcal{A}$ is special and that, by construction, $l_{\mathcal{A}}$ is of type $pI_d - \mathcal{A}$. Moreover, $\mathbf{l}$ is also of type $pI_d - \mathcal{A}$ because of Equation (2.4).
Furthermore, Honda showed in [Hon70, Theorem 2] that two formal groups with vectorial logarithms of the same type are isomorphic over $W$. Thus, we can replace the study of the formal group $G$ by the one of $F$, which is easier because we know an explicit expression of its logarithms, which gives us a control on denominators.

## 2.2 Properties of the formal group $F$

In this section, the reader can refer to [Abr97] from which we recall principal constructions.

Let us first describe the Dieudonné module of $F$.
We already know a basis of the logarithms, the coordinate power series of the vectorial series

$$l_{\mathcal{A}}(\mathbf{X}) = \mathbf{X} + \sum_{m\geq 1} \frac{\mathcal{A}^m(\mathbf{X})}{p^m}.$$

Complete it into a basis of $H^1(F)$ by putting

$$m_{\mathcal{A}}(\mathbf{X}) = \sum_{u\geq 1} F'_u \frac{\varphi^u(l_{\mathcal{A}}(\mathbf{X}))}{p}.$$

Let $o = (o_s)_{s\geq 0} \in T(F)$. For all $s \geq 0$, choose a lift $\hat{o}_s \in W(\mathfrak{m}_{\tilde{\mathbf{E}}})^d$ of $o_s$, that is, with $\theta(\hat{o}_s) = o_s$. Then the following lemma says that the sequence $p^s \mathrm{id}_F \hat{o}_s$ converges in $W^1(\mathfrak{m}_{\tilde{\mathbf{E}}})^d$ towards an element $j(o)$ independent of the choice of lifts:

**Lemma 2.1.**

1. *The series $l_{\mathcal{A}}$ defines a continuous one-to-one homomorphism of $G_K$-modules*

$$l_{\mathcal{A}} \ : \ F(W(\mathfrak{m}_{\tilde{\mathbf{E}}})) \to A_{crys}^d \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

   *Moreover, the restriction of $l_{\mathcal{A}}$ to $F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}}))$ takes values in* $(\mathrm{Fil}^1 A_{crys})^d$.

2. *The endomorphism $p\mathrm{id}_F$ of $F(W(\mathfrak{m}_{\tilde{\mathbf{E}}}))$ is topologically nilpotent. The convergence of $p\mathrm{id}_F$ to zero is uniform on $F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}}))$.*

3. *The map* $j : T(F) \to W^1(\mathfrak{m}_{\tilde{\mathbf{E}}})^d$ *is well defined and provides a continuous one-to-one homomorphism of* $G_K$*-modules* $j : T(F) \to F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}}))$.

*Proof:* Point 1. is Lemma 1.5.1 of [Abr97].
Point 2. follows from the fact that $W^1(\mathfrak{m}_{\tilde{\mathbf{E}}}) = \omega W(\mathfrak{m}_{\tilde{\mathbf{E}}})$ with $\omega = X/\varphi^{-1}(X) \in W(\mathfrak{m}_{\tilde{\mathbf{E}}}) + p\tilde{\mathbf{A}}^+$ and that the series corresponding to $p\mathrm{id}_F$ can be written as

$$p\mathrm{id}_F\mathbf{X} = p\mathbf{X} + \text{higher degrees}.$$

Let us recall briefly the proof of Point 3.:
For all $s \geq 0$,

$$\theta(p^s\mathrm{id}_F\hat{o}_s) = o_0 = 0$$

so that $p^s\mathrm{id}_F\hat{o}_s \in F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}}))$. On the other hand, for all $s \geq 0$,

$$p\mathrm{id}_F\hat{o}_{s+1} \equiv \hat{o}_s \mod F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}}))$$

thus

$$p^{s+1}\mathrm{id}_F\hat{o}_{s+1} \equiv p^s\mathrm{id}_F\hat{o}_s \mod p^s\mathrm{id}_F\left(F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}}))\right)$$

And Point 2. provides the convergence of the sequence $(p^s\mathrm{id}_F\hat{o}_s)_s$.
The fact that the convergence is given without compatibility condition on the lifts shows the independence of the limit with respect to the choice of these lifts. Namely, let $(\hat{o}_s)_{s\geq 0}$ and $(\hat{o}'_s)_{s\geq 0}$ be two given lifts of $(o_s)_{s\geq 0}$, then for any lift $(\hat{o}''_s)_{s\geq 0}$ where

$$\forall s \geq 0,\ \hat{o}''_s = \hat{o}_s \text{ or } \hat{o}'_s,$$

we still have the convergence of $(p^s\mathrm{id}_F\hat{o}''_s)_s$, from which we deduce that the limits are the same.
The remainder is straightforward. □

Composing the vectorial logarithm $l_{\mathcal{A}}$ with $j$ gives a $G_K$-equivariant injection that we will denote by $\mathbf{l}$ from $T(F)$ into $(\mathrm{Fil}^1 A_{crys})^d$. This map satisfies then for any $o$ in $T(F)$:

$$\mathbf{l}(o) = l_{\mathcal{A}}(\lim_{s\to\infty} p^s\mathrm{id}_F\hat{o}_s) = \lim_{s\to\infty} p^s l_{\mathcal{A}}(\hat{o}_s).$$

Put now

$$\mathbf{m} = \sum_{u\geq 1} F'_u \frac{\varphi^u(\mathbf{l})}{p},$$

then $\begin{pmatrix}\mathbf{l}\\ \mathbf{m}\end{pmatrix}$ provides a basis of $D^0(F)$ with $\mathbf{l}$ a basis of $D^1(F)$. The map

$$\begin{pmatrix}\mathbf{l}\\ \mathbf{m}\end{pmatrix} : T(F) \to A^h_{crys}$$

then factorizes through

$$\begin{pmatrix} l_{\mathcal{A}} \\ m_{\mathcal{A}} \end{pmatrix} \; : \; F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}})) \to A^h_{crys}.$$

This map is the period pairing. Recall (cf. [Abr97], Remark 1.7.5) that this map takes values in $\tilde{\mathbf{A}}^+[[X^{p-1}/p]]$. It is also a consequence of Wach's computation for potentially crystalline representations (cf. [Wac96]).

Fix now a basis $(o^1, \ldots, o^h)$ of $T(F)$. We can then introduce the period matrix

$$\mathcal{V} = \begin{pmatrix} \mathbf{l}(o^1) & \ldots & \mathbf{l}(o^h) \\ \mathbf{m}(o^1) & \ldots & \mathbf{m}(o^h) \end{pmatrix} \in M_h(\tilde{\mathbf{A}}^+[[X^{p-1}/p]]) \cap GL_h(\mathrm{Frac}\tilde{\mathbf{A}}^+[[X^{p-1}/p]]).$$

It satisfies

$$\begin{pmatrix} I_d \frac{\varphi}{p} & 0 \\ 0 & I_{h-d}\varphi \end{pmatrix} \mathcal{V} = \mathcal{E}\mathcal{V}.$$

Remark that the inverse of $\mathcal{V}$ is then the change of basis matrix from the basis $(o^1, \ldots, o^h)$ to a basis of

$$D_{crys}(T(F)) = (T(F) \otimes_{\mathbb{Z}_p} A_{crys})^{G_{K_0}},$$

the covariant version of the crystalline module of Fontaine's theory associated with $T(F)$.

Let $u \in T(F) \otimes A_{crys}$, and $U$ be the coordinate vector of $u$ in the basis $(o^1, \ldots, o^h)\mathcal{V}^{-1}$ of $D_{crys}(T(F))$, then we can compute the coordinates of

$$\varphi(u) = (o^1, \ldots, o^h)\varphi(\mathcal{V}^{-1})\varphi(U).$$

We know that

$$\varphi(\mathcal{V}) = \begin{pmatrix} pI_d & 0 \\ 0 & I_{h-d} \end{pmatrix} \begin{pmatrix} I_d \frac{\varphi}{p} & 0 \\ 0 & I_{h-d}\varphi \end{pmatrix} \mathcal{V} = \begin{pmatrix} pI_d & 0 \\ 0 & I_{h-d} \end{pmatrix} \mathcal{E}\mathcal{V}$$

so that

$$\varphi(\mathcal{V}^{-1}) = \mathcal{V}^{-1}\mathcal{E}^{-1} \begin{pmatrix} p^{-1}I_d & 0 \\ 0 & I_{h-d} \end{pmatrix}$$

and coordinates of $\varphi(y)$ in the basis $(o^1, \ldots, o^h)\mathcal{V}^{-1}$ are then

$$\mathcal{E}^{-1} \begin{pmatrix} I_d \frac{\varphi}{p} & 0 \\ 0 & I_{h-d}\varphi \end{pmatrix} U.$$

Keeping this in mind, the following lemma shows that $\begin{pmatrix} \frac{\mathcal{A}}{p} & 0 \\ 0 & I_{h-d} \end{pmatrix}$ acts as the Frobenius on $D_{crys}(T(F))$.

**Lemma 2.2.**
*The following equality holds:*

$$\mathcal{E}^{-1}\begin{pmatrix} \frac{\varphi}{p}\circ l_{\mathcal{A}} \\ \varphi\circ m_{\mathcal{A}} \end{pmatrix} = \begin{pmatrix} \frac{\mathcal{A}}{p}\circ l_{\mathcal{A}} \\ m_{\mathcal{A}} \end{pmatrix}$$

*Proof:* Compute:

$$A\frac{\varphi}{p}(l_{\mathcal{A}}) + B\varphi(m_{\mathcal{A}}) = A\frac{\varphi}{p}(l_{\mathcal{A}}) + \sum_{u\geq 1} B\varphi F'_u \frac{\varphi^u(l_{\mathcal{A}})}{p} = \frac{\mathcal{A}}{p}(l_{\mathcal{A}})$$

for $B\varphi F'_u = F_{u+1}$ for all $u \geq 1$. And:

$$C\frac{\varphi}{p}(l_{\mathcal{A}}) + D\varphi(m_{\mathcal{A}}) = C\frac{\varphi}{p}(l_{\mathcal{A}}) + \sum_{u\geq 1} D\varphi F'_u \frac{\varphi^u(l_{\mathcal{A}})}{p} = m_{\mathcal{A}}$$

since $D\varphi F'_u = F'_{u+1}$ for all $u \geq 1$. $\square$

Abrashkin also computed the cokernel of injection $j$ (cf. [Abr97, Proposition 2.1.]):

**Proposition 2.1.**
*There is an equality*

$$(\mathcal{A}-p)\circ l_{\mathcal{A}}(F(W(\mathfrak{m}_{\tilde{\mathbf{E}}}))) = (\mathcal{A}-p)\circ l_{\mathcal{A}}(F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}})))$$

*and the following sequence is exact:*

$$0 \longrightarrow T(F) \xrightarrow{j} F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}})) \xrightarrow{\left(\frac{\mathcal{A}}{p}-1\right)\circ l_{\mathcal{A}}} W(\mathfrak{m}_{\tilde{\mathbf{E}}})^d \longrightarrow 0$$

**Remark** Beware that if $x \in F(W(\mathfrak{m}_{\tilde{\mathbf{E}}}))$,

$$\varphi(l_{\mathcal{A}})(x) = \varphi(l_{\mathcal{A}}(x))$$

and then

$$\mathcal{A}(l_{\mathcal{A}})(x) = \mathcal{A}(l_{\mathcal{A}}(x))$$

hold if $\varphi(x) = x^p$ (e.g. when $x$ is a Teichmüller representative) but not in general ! On the left side, $\varphi$ and $\mathcal{A}$ act on $W[[\mathbf{X}]]$, whereas they act on $A_{crys}$ on the right side.

Abrashkin showed furthermore (cf. [Abr97, Lemma 1.6.2.])

**Lemma 2.3.**
*$F(\mathfrak{m}_{\tilde{\mathbf{E}}})$ is uniquely $p$-divisible.*

This provides a continuous one-to-one $G_K$-equivariant morphism

$$\delta \ : \ F(\mathfrak{m}_{\tilde{\mathbf{E}}}) \to F(W(\mathfrak{m}_{\tilde{\mathbf{E}}}))^{(\mathcal{A}-p)\circ l_{\mathcal{A}}=0}$$

defined as follows: let $x \in F(\mathfrak{m}_{\tilde{\mathbf{E}}})$, then because of the lemma, for all $s \geq 0$ there exists a unique $x_s \in F(\mathfrak{m}_{\tilde{\mathbf{E}}})$ such that

$$p^s \mathrm{id}_F x_s = x.$$

Thus the sequence $(p^s \mathrm{id}_F[x_s])_s$ converges to an element $\delta(x)$ in $F(W(\mathfrak{m}_{\tilde{\mathbf{E}}}))$.
$\delta$ is a morphism since

$$\delta(x +_F y) = \lim_s p^s \mathrm{id}_F[x_s +_F y_s] = \lim_s p^s \mathrm{id}_F([x_s] +_F [y_s] +_F u_s)$$

with $u_s \in pW(\mathfrak{m}_{\tilde{\mathbf{E}}})$ where the convergence of $p^s \mathrm{id}_F$ towards zero is uniform.
Moreover, for $\mathcal{A} \circ l_{\mathcal{A}}$ coincides with $\mathcal{A}(l_{\mathcal{A}})$ on Teichmüller representatives, we get the last point:

$$(\mathcal{A} - p) \circ l_{\mathcal{A}}(\delta(x)) = (\mathcal{A} - p)(l_{\mathcal{A}})(\delta(x)) = 0.$$

Finally, remark

$$\theta(\delta(x)) = \theta([x]).$$

Namely, for all $s \geq 0$,

$$\theta(p^s \mathrm{id}_F[x_s]) = p^s \mathrm{id}_F \theta([x_s]) = \theta([x]).$$

## 2.3 The ring $\mathcal{G}_{[b,a]}$ and some subrings.

### 2.3.1 Introducing the objects

Fix $e$ the absolute ramification index of $K$.
In [Ber02], Berger introduced for $s \geq r \geq 0$ the ring $\tilde{\mathbf{A}}_{[s,r]}$, the $p$-adic completion of the ring

$$\tilde{\mathbf{A}}^+ \left[ \frac{p}{Y^{rep/(p-1)}}, \frac{Y^{sep/(p-1)}}{p} \right].$$

Let us then introduce for $a > b \geq 0$, the ring

$$\mathcal{G}_{[b,a]} := \tilde{\mathbf{A}}^+ \left[ \left[ \frac{Y^{ae}}{p}, \frac{p}{Y^{be}} \right] \right]$$

which for integers $a$ and $b$ admits the description

$$\mathcal{G}_{[b,a]} = \left\{ \sum_{n \in \mathbb{Z}} a_n Y^n \mid a_n \in \tilde{\mathbf{A}}^+ \left[ \frac{1}{p} \right], \begin{array}{ll} aev_p(a_n) + n \geq 0 & \text{for } n \geq 0 \\ bev_p(a_n) + n \geq 0 & \text{for } n \leq 0 \end{array} \right\}.$$

Note that the expression $\sum_{n \in \mathbb{Z}} a_n Y^n$ for an element of $\mathcal{G}_{[b,a]}$ is not unique. The ring $\mathcal{G}_{[b,a]}$ is naturally, for $a > \alpha \geq \beta > b$ a subring of $\tilde{\mathbf{A}}_{[\alpha(p-1)/p, \beta(p-1)/p]}$. We even have inclusions

$$\tilde{\mathbf{A}}_{[a(p-1)/p, b(p-1)/p]} \subset \mathcal{G}_{[b,a]} \subset \tilde{\mathbf{A}}_{[\alpha(p-1)/p, \beta(p-1)/p]}.$$

Endow then $\mathcal{G}_{[b,a]}$ with the induced topology, which is well defined since inclusions

$$\tilde{\mathbf{A}}_{[r_1,s_1]} \hookrightarrow \tilde{\mathbf{A}}_{[r_2,s_2]}$$

for $r_1 \leq r_2 \leq s_2 \leq s_1$ are continuous.
This topology then admits as a basis of neighborhoods of zero

$$\left\{ \left\{ \sum_{n>N} a_n \left( \frac{Y^{ae}}{p} \right)^n + \sum_{n>N} b_n \left( \frac{p}{Y^{be}} \right)^n ; a_n, b_n \in \tilde{\mathbf{A}}^+ \right\} + p^k \mathcal{G}_{[b,a]} \right\}_{N,k \in \mathbb{N}}.$$

For $a_1 > a_2 > b_2 > b_1 \geq 0$ we still have continuous injections

$$\mathcal{G}_{[a_1,b_1]} \hookrightarrow \mathcal{G}_{[a_2,b_2]}.$$

Denote then for $a \geq b \geq 0$ by $\mathcal{G}_{[b,a[}$ the $p$-adic completion of $\bigcup_{\alpha>a} \mathcal{G}_{[b,\alpha]}$. For integers $a$ and $b$, it admits the description:

$$\mathcal{G}_{[b,a[} = \left\{ \sum_{n\in\mathbb{Z}} a_n Y^n ; a_n \in \tilde{\mathbf{A}}^+ \left[ \frac{1}{p} \right], \begin{array}{ll} aev_p(a_n) + n > 0 & \text{for } n > 0 \text{ and} \\ aev_p(a_n) + n \underset{n\to+\infty}{\longrightarrow} +\infty, & \\ bev_p(a_n) + n \geq 0 & \text{for } n \leq 0 \end{array} \right\}.$$

Because of the inclusion

$$\bigcup_{\alpha>a} \mathcal{G}_{[b,\alpha]} \hookrightarrow \bigcup_{\alpha>a,\beta>b} \tilde{\mathbf{A}}_{[\alpha(p-1)/p,\beta(p-1)/p]}$$

we endow $\mathcal{G}_{[b,a[}$ with the topology induced by the $p$-adic toplogy of the $p$-adic completion of $\bigcup_{\alpha>a,\beta>b} \tilde{\mathbf{A}}_{[\alpha(p-1)/p,\beta(p-1)/p]}$.
Let us also introduce for $b \geq 0$,

$$\mathcal{G}_{[b,\infty[} := \bigcap_{a>b} \mathcal{G}_{[b,a]} = \tilde{\mathbf{A}}^+ \left[ \left[ \frac{p}{Y^{eb}} \right] \right] \subset \tilde{\mathbf{A}}$$

which is for $b$ integer

$$\mathcal{G}_{[b,\infty[} = \left\{ \sum_{n\leq 0} a_n Y^n ; a_n \in \tilde{\mathbf{A}}^+, bev_p(a_n) + n \geq 0 \text{ for } n \leq 0 \right\}.$$

Remark that the Frobenius

$$\varphi_{\mathcal{G}} \left( \sum_{n<0} a_n Y^{aen} + \sum_{n\geq 0} a_n Y^{ben} \right) = \sum_{n<0} \varphi(a_n) Y^{paen} + \sum_{n\geq 0} \varphi(a_n) Y^{pben}$$

defines a one-to-one morphism from $\mathcal{G}_{[b,a]}$ (respectively $\mathcal{G}_{[b,a[}$) into $\mathcal{G}_{[pb,pa]}$ (respectively $\mathcal{G}_{[pb,pa[}$).

Introduce for integers $a$ and $b$ the subring of $\mathcal{G}_{[b,a]}$:

$$\begin{aligned}\mathcal{G}_{Y,[b,a]} &:= W[[Y]]\left[\left[\frac{Y^{ae}}{p}, \frac{p}{Y^{be}}\right]\right] \\ &= \left\{\sum_{n\in\mathbb{Z}} a_n Y^n \ ;\ a_n \in K_0,\ \begin{array}{ll} aev_p(a_n)+n \geq 0 & \text{for } n \geq 0 \\ bev_p(a_n)+n \geq 0 & \text{for } n \leq 0 \end{array}\right\}\end{aligned}$$

and $\mathcal{G}_{Y,[b,a[}$ the subring of $\mathcal{G}_{[b,a[}$ admitting the description

$$\mathcal{G}_{Y,[b,a[} = \left\{\sum_{n\in\mathbb{Z}} a_n Y^n \ ;\ a_n \in K_0,\ \begin{array}{ll} aev_p(a_n)+n > 0 & \text{for } n > 0 \text{ and} \\ aev_p(a_n)+n \underset{n\to+\infty}{\longrightarrow} +\infty, & \\ bev_p(a_n)+n \geq 0 & \text{for } n \leq 0 \end{array}\right\}.$$

Finally, for $b \geq 0$,

$$\begin{aligned}\mathcal{G}_{Y,[b,\infty[} &:= \bigcap_{a>b} \mathcal{G}_{Y,[b,a]} = W[[Y]]\left[\left[\frac{p}{Y^{eb}}\right]\right] \\ &= \left\{\sum_{n\leq 0} a_n Y^n ; a_n \in K_0, bev_p(a_n)+n \geq 0\right\}.\end{aligned}$$

Contrary to the above situation, the expression $\sum_{n\in\mathbb{Z}} a_n Y^n$ is unique as it is shown in the following

**Lemma 2.4.**

1. *In* $\mathcal{G}_{[b,a]}\left[\frac{1}{p}\right]$, *one has* $\mathcal{G}_{[0,a]}\left[\frac{1}{p}\right] \bigcap \mathcal{G}_{[b,\infty[}\left[\frac{1}{Y}\right] = \tilde{\mathbf{A}}^+$.

2. *An element of* $\mathcal{G}_{Y,[b,a]}$ *or* $\mathcal{G}_{Y,[b,a[}$ *can uniquely be written as* $\sum_{n\in\mathbb{Z}} a_n Y^n$ *with* $a_n \in K_0$.

3. *For* $a \geq \alpha \geq \beta \geq b$, *and* $)$ *designating* $]$ *or* $[$, *one has*

$$\mathcal{G}_{Y,[\beta,\alpha]}\left[\frac{1}{p}\right] \bigcap \mathcal{G}_{[b,a)} = \mathcal{G}_{Y,[b,a)}.$$

*Proof:* The first point can be shown in Berger's rings $\tilde{\mathbf{A}}_{[s,r]}$, in fact in the ring $\tilde{\mathbf{A}}_{[s,\infty[}\left[\frac{1}{Y}\right] + \tilde{\mathbf{A}}_{[0,r]}\left[\frac{1}{p}\right]$. Any element of this ring is of the form $\sum_{n\in\mathbb{N}} p^n(\frac{x_n}{Y^k} - \frac{y_n}{p^l})$ with $x_n \in \tilde{\mathbf{A}}^+\left[\frac{p}{Y^{rep/(p-1)}}\right]$ and $y_n \in \tilde{\mathbf{A}}^+\left[\frac{Y^{sep/(p-1)}}{p}\right]$. Such an element is zero when

$$p^l \sum_{n\in\mathbb{N}} p^n x_n = Y^k \sum_{n\in\mathbb{N}} p^n y_n \in \tilde{\mathbf{A}}_{[s,\infty[} \bigcap \tilde{\mathbf{A}}_{[0,r]}.$$

The condition is that for all $N \in \mathbb{N}$,

$$\sum_{n<N} p^n(p^l x_n - Y^k y_n) \in p^N \tilde{\mathbf{A}}_{[s,r]}.$$

That is

$$p^l \sum_{n<N} p^n x_n \in \tilde{\mathbf{A}}^+ + p^N \tilde{\mathbf{A}}^+ \left[\frac{p}{Y^{rep/(p-1)}}\right]$$

and then

$$\sum_{n<N} p^n x_n \in \tilde{\mathbf{A}}^+ + p^{N-l} \tilde{\mathbf{A}}^+ \left[\frac{p}{Y^{rep/(p-1)}}\right],$$

and similarly

$$\sum_{n<N} p^n y_n \in \tilde{\mathbf{A}}^+ + p^{N-l/se} \tilde{\mathbf{A}}^+ \left[\frac{Y^{sep/(p-1)}}{p}\right].$$

Thus the limit $p^l \sum_{n\in\mathbb{N}} p^n x_n = Y^k \sum_{n\in\mathbb{N}} p^n y_n$ lies in $p^l \tilde{\mathbf{A}}^+ \bigcap Y^k \tilde{\mathbf{A}}^+ = p^l Y^k \tilde{\mathbf{A}}^+$, hence

$$\sum_{n\in\mathbb{N}} p^n \frac{x_n}{Y^k} = \sum_{n\in\mathbb{N}} p^n \frac{y_n}{p^l} \in \tilde{\mathbf{A}}^+$$

as claimed.

Because of the first point, it is enough to prove the second one for $\sum_{n<0} a_n Y^n$ and $\sum_{n>0} a_n Y^n$. It is to prove that such a series converges to zero if and only if all the $a_n$ actually are zero. In the first case, it is a series converging in $\tilde{\mathbf{A}}$ and the natural map $\mathcal{G}_{Y,[\beta,\infty[} \to \tilde{\mathbf{A}}$ is a continuous one-to-one morphism. Successive approximations modulo $p^n$ and modulo $Y^k$ then provide the result. On the other side, $\mathcal{G}_{Y,[0,\alpha)}$ is naturally a subring of the separable completion of $\tilde{\mathbf{A}}\left[\frac{1}{p}\right]$ for the $Y$-adic topology. The result then follows from successive reductions modulo $Y^k$.

We use similar techniques to show the last point. Again, because of the first one, it suffices to prove both $\mathcal{G}_{Y,[\beta,\infty[}\left[\frac{1}{p}\right] \bigcap \mathcal{G}_{[b,\infty[} = \mathcal{G}_{Y,[b,\infty[}$ and $\mathcal{G}_{Y,[0,\alpha]}\left[\frac{1}{p}\right] \bigcap \mathcal{G}_{[0,a)} = \mathcal{G}_{Y,[0,a)}$. First consider then $x = \sum_{n\leq 0} a_n Y^n \in \frac{1}{p^\lambda}\mathcal{G}_{Y,[\beta,\infty[}$ with $\beta ev_p(a_n) + n + \lambda \geq 0$ for all $n$. We suppose furthermore that $x$ belongs to $\mathcal{G}_{[b,\infty[}$, that is, it can be written as $\sum_{n\in\mathbb{N}} b_n \frac{p^n}{Y^{ebn}}$ with $b_n \in \tilde{\mathbf{A}}^+$. The identity

$$\sum_{n\leq 0} a_n Y^n = \sum_{n\in\mathbb{N}} b_n \frac{p^n}{Y^{ebn}}$$

makes sense in $\frac{1}{p^\lambda}\mathcal{G}_{[\beta,\infty[}$, thus in $\tilde{\mathbf{A}}$. Denote by $n_0$ the highest integer, supposing it exists, satisfying $bev_p(a_{n_0}) + n_0 < 0$. We can then suppose that the identity above is of the form

$$\sum_{n\leq n_0} a_n Y^n = \sum_{n\in\mathbb{N}} b_n \frac{p^n}{Y^{ebn}}.$$

Multiplying by $Y^{bev_p(a_{n_0})}$ and reducing modulo $p^{v_p(a_{n_0})}$ yields then to

$$\sum_{n=n_0'}^{n_0} a_n Y^{n+bev_p(a_{n_0})} \equiv \sum_{n=0}^{v_p(a_{n_0})} b_n p^n Y^{eb(v_p(a_{n_0})-n)} \mod p^{v_p(a_{n_0})}$$

but the right term is entire (it belongs to $\tilde{\mathbf{A}}^+$) and not the left one, whence a contradiction.

Now, consider as before an identity of the form

$$\sum_{n\geq 0} a_n Y^n = \sum_{n\in\mathbb{N}} b_n \frac{Y^{ean}}{p^n}$$

and denote by $n_0$ the lowest integer satisfying $aev_p(a_{n_0}) + n_0 < 0$. It can be reduced to an identity of the form

$$\sum_{n\geq n_0} a_n Y^n = \sum_{n\in\mathbb{N}} b_n \frac{Y^{ean}}{p^n}.$$

Multiplying by $p^{-v_p(a_{n_0})}$ and reducing modulo $Y^{n_0+1}$ yields to

$$p^{-v_p(a_{n_0})} a_{n_0} Y^{n_0} \equiv \sum_{0\leq n\leq \frac{n_0}{ea}} b_n p^{-v_p(a_{n_0})-n} Y^{ean} \mod Y^{n_0+1}$$

and the contradiction comes from the inequality

$$n \leq \frac{n_0}{ea} < -v_p(a_{n_0})$$

hence the right term is divisible by $p$, and not the left one.

The case of $\mathcal{G}_{Y,[0,\alpha]} \bigcap \mathcal{G}_{[0,a[} = \mathcal{G}_{Y,[0,a[}$ follows from a similar argument. $\square$

**Remark** As said before, periods of formal groups lie in the ring

$$\tilde{\mathbf{A}}^+[[X^{p-1}/p]] = \tilde{\mathbf{A}}^+[[Y^{pe}/p]] = \mathcal{G}_{[0,p]}.$$

We can also recover some well known rings by

$$\begin{aligned} \tilde{\mathbf{A}}^+ &= \mathcal{G}_{[0,\infty[} \\ \mathbf{B}^\dagger_{rig} &= \bigcup_{b>0} \mathcal{G}_{[b,\infty[}. \end{aligned}$$

### 2.3.2 Some topological precisions

**Lemma 2.5.**

1. *Finite sums*

$$\left\{ \sum_{n=0}^{N} a_n \left(\frac{Y^{ea}}{p}\right)^n + b_n \left(\frac{p}{Y^{eb}}\right)^n ; a_n, b_n \in \tilde{\mathbf{A}}^+, N \in \mathbb{N} \right\}$$

*form a dense subset of $\mathcal{G}_{[b,a]}$. The same holds for the sub-algebra*

$$\mathcal{G}_{[b,a]} \bigcap \tilde{\mathbf{A}}\left[\frac{1}{p}\right] = \left\{ \sum_{n=0}^{N} a_n \left(\frac{Y^{ea}}{p}\right)^n + \sum_{n\in\mathbb{N}} b_n \left(\frac{p}{Y^{eb}}\right)^n ; a_n, b_n \in \tilde{\mathbf{A}}^+, N \in \mathbb{N} \right\}.$$

2. *The topology of $\mathcal{G}_{[b,a]}$ is weaker than the p-adic topology.*

3. *$\mathcal{G}_{[b,a]}$ is Hausdorff and complete.*

4. *The ring $\mathcal{G}_{[b,a]}$ is local with maximal ideal*

$$\mathfrak{m}_{[b,a]} = \left\{ \sum_{n>1} a_n \left(\frac{Y^{ea}}{p}\right)^n + b_n \left(\frac{p}{Y^{eb}}\right)^n ; a_n, b_n \in \tilde{\mathbf{A}}^+ \right\} + W(\mathfrak{m}_{\tilde{\mathbf{E}}})$$

*and residue $\overline{k}$.*

5. *Any element of $\mathfrak{m}_{[b,a]}$ is topologically nilpotent.*

6. *Powers of the ideal*

$$\mathfrak{m}^1_{[b,a]} = \left\{ \sum_{n>1} a_n \left(\frac{Y^{ea}}{p}\right)^n + b_n \left(\frac{p}{Y^{eb}}\right)^n ; a_n, b_n \in \tilde{\mathbf{A}}^+ \right\} + Y^{e(a-b)}\tilde{\mathbf{A}}^+$$

*form a basis of neighborhoods of $0$ consisting in ideals of $\mathcal{G}_{[b,a]}$.*

7. *The ring $\mathcal{G}_{[b,a[}$ is local with maximal ideal $\mathfrak{m}_{[b,a[}$ the p-adic completion of*

$$\bigcup_{\alpha>a} \mathfrak{m}_{[b,\alpha]}$$

*and with residue field $\overline{k}$.*

8. *Any element of $\mathfrak{m}_{[b,a[}$ is topologically nilpotent.*

*Proof:* Let us introduce the notation

$$\mathcal{G}^{>N}_{[b,a]} = \left\{ \sum_{n>N} a_n \left(\frac{Y^{ae}}{p}\right)^n + \sum_{n>N} b_n \left(\frac{p}{Y^{be}}\right)^n ; a_n, b_n \in \tilde{\mathbf{A}}^+ \right\} \subset \mathcal{G}_{[b,a]}.$$

Recall that a basis of neighborhoods of zero in $\mathcal{G}_{[b,a]}$ is given by

$$\left\{ \mathcal{G}^{>N}_{[b,a]} + p^k \mathcal{G}_{[b,a]} \right\}_{N,k\in\mathbb{N}}.$$

This shows the first two points. The fact that $\mathcal{G}_{[b,a]}$ is Hausdorff follows from that $\tilde{\mathbf{A}}_{[s,r]}$ is (cf. [Ber02]).
The following shows that the topology on $\mathcal{G}_{[b,a]}$ is metrizable, and one can immediately see from the form of neighborhoods of zero that any series with a general term going to 0 converges. This shows that $\mathcal{G}_{[b,a]}$ is complete.
We will prove Points 4., 5. et 6. simultaneously: we first show $\mathfrak{m}_{[b,a]}$ is an ideal, then that any element of $\mathfrak{m}_{[b,a]}$ has a power belonging to $\mathfrak{m}^1_{[b,a]}$ and we make powers of $\mathfrak{m}^1_{[b,a]}$ explicit, which allows to conclude.

Let

$$x = \sum_{n<0} a_n \left(\frac{Y^{ea}}{p}\right)^{-n} + \sum_{n\geq 0} a_n \left(\frac{p}{Y^{eb}}\right)^n$$

we say that $x$ is the element of $\mathcal{G}_{[b,a]}$ associated with the sequence $(a_n)_{n\in\mathbb{Z}} \in \left(\tilde{\mathbf{A}}^+\right)^{\mathbb{Z}}$. Let $y$ be the element associated with another sequence $(b_n)_{n\in\mathbb{Z}}$, then write the product of two elements $x$ and $y$ of $\mathcal{G}_{[b,a]}$:

$$xy = \sum_{n<0} c_n \left(\frac{Y^{ea}}{p}\right)^{-n} + \sum_{n\geq 0} c_n \left(\frac{p}{Y^{eb}}\right)^n$$

is associated with the sequence

$$c_n = \begin{cases} \sum_{k>0} Y^{e(a-b)k}(a_{k+n}b_{-k} + a_{-k}b_{k+n}) + \sum_{k=0}^{n} a_k b_{n-k} & \text{if } n \geq 0 \\ \sum_{k>0} Y^{e(a-b)k}(a_k b_{n-k} + a_{n-k}b_k) + \sum_{k=0}^{-n} a_{-k} b_{n+k} & \text{if } n \leq 0. \end{cases} \tag{2.5}$$

This yields to

$$c_0 = \sum_{n\in\mathbb{Z}} Y^{e(a-b)|n|} a_n b_{-n} \tag{2.6}$$

and shows that $\mathfrak{m}_{[b,a]}$ is an ideal.
Suppose $x \in \mathfrak{m}_{[b,a]}$. Because of the previous computation, one can define for all $k \in \mathbb{N}$ a sequence $(c_{n,k})_{n\in\mathbb{Z}}$ such that $x^k$ is associated with $(c_{n,k})_{n\in\mathbb{Z}}$. The fact that there exists a $k$ such that $x^k \in \mathfrak{m}^1_{[b,a]}$ is equivalent to that the rest $\overline{c}_{0,k} \in \tilde{\mathbf{E}}^+$ of $c_{0,k}$ modulo $p$ has a valuation greater or equal to $a-b$. But because of Equality (2.6),

$$v_{\mathbf{E}}(\overline{c}_{0,k}) \geq \min(a-b, k v_{\mathbf{E}}(\overline{a}_0))$$

which shows $x^k \in \mathfrak{m}^1_{[b,a]}$ for $k$ large enough.
Let us show now that $\mathfrak{m}^k_{[b,a]}$ consists in elements associated with sequences $(a_n)_{n\in\mathbb{Z}}$ such that

$$\forall n \in \mathbb{Z},\ v_{\mathbf{E}}(\overline{a}_n) \geq g^k_{a,b}(n)$$

where

$$g^k_{a,b}(n) = \left\lfloor \frac{(k-|n|+1)_+}{2} \right\rfloor (a-b) = \begin{cases} \left\lfloor \frac{k-|n|+1}{2} \right\rfloor (a-b) & \text{if } |n| \leq k \\ 0 & \text{otherwise} \end{cases}$$

satisfying the induction relation

$$g^{k+1}_{a,b}(n) = \begin{cases} g^k_{a,b}(n-1) + a - b & \text{if } -k-1 \leq n \leq 0 \\ g^k_{a,b}(n+1) + a - b & \text{if } 0 \leq n \leq k+1 \\ 0 & \text{otherwise} \end{cases} \tag{2.7}$$

or equivalently

$$g_{a,b}^{k+1}(n) = \begin{cases} g_{a,b}^{k}(n+1) & \text{if } n < 0 \\ g_{a,b}^{k}(n-1) & \text{if } n > 0 \end{cases}. \tag{2.8}$$

Remark also that $g_{a,b}^k$ is even and decreasing on $\mathbb{N}$.
Let then $x \in \mathfrak{m}_{[b,a]}^k$ be associated with a sequence $(a_n)_{n\in\mathbb{Z}}$ satisfying the previous induction relation, let $y \in \mathfrak{m}_{[b,a]}^k$ be associated with $(b_n)_{n\in\mathbb{Z}}$ and $xy \in \mathfrak{m}_{[b,a]}^{k+1}$ be associated with $(c_n)_{n\in\mathbb{Z}}$ which we compute as before.
Equations (2.5) show the relation for $n \geq 0$ (case $n < 0$ provides the same computation):

$$v_{\mathbf{E}}(\overline{c}_n) \geq \inf \left\{ \begin{array}{ll} (a-b)r + g_{a,b}^k(n+r), & \text{for } r > 0, \\ (a-b)r + g_{a,b}^k(-r), & \text{for } r > 0, \\ g_{a,b}^k(r), & \text{for } 0 \leq r < n, \\ g_{a,b}^k(n) + a - b & \end{array} \right\}$$

which gives because $g_{a,b}^k$ is even and decreasing

$$v_{\mathbf{E}}(\overline{c}_n) \geq \inf \left\{ \begin{array}{ll} (a-b)r + g_{a,b}^k(n+r), & \text{for } r > 0, \\ g_{a,b}^k(n-1) & \\ g_{a,b}^k(n) + a - b & \end{array} \right\}.$$

But

$$(a-b)r + g_{a,b}^k(n+r) = (a-b)\left(r + \left\lfloor \frac{(k - |n+r| + 1)_+}{2} \right\rfloor\right)$$

is strictly increasing in $r$ and

$$(a-b) + g_{a,b}^k(n+1) \geq g_{a,b}^{k+1}(n)$$

because of (2.7). Likewise,

$$g_{a,b}^k(n) + a - b \geq g_{a,b}^{k+1}(n-1) \geq g_{a,b}^{k+1}(n)$$

and finally, according to (2.8),

$$g_{a,b}^k(n-1) = g_{a,b}^{k+1}(n).$$

The minimum is then equal to $g_{a,b}^{k+1}(n)$, which lets us conclude on the description of $\mathfrak{m}_{[b,a]}^k$.
Point 6. follows from this description, and proves 5. and 4.
At last, 7. is a consequence of 8., which is left to show. Remark that any element $x \in \mathfrak{m}_{[b,a[}$ can be written as

$$x = x_0 + px_1 \ ; \ x_0 \in \mathfrak{m}_{[b,\alpha]}, x_1 \in \mathcal{G}_{[b,a[}$$

for some $\alpha > a$. We have to show

$$p^k x_0^n x_1^k \underset{k,n\to+\infty}{\longrightarrow} 0.$$

When $k$ goes to infinity, it is clear. For the case where $n$ goes to infinity, remark that the convergence of $x_0^n$ to 0 in $\mathcal{G}_{[b,\alpha]}$ implies for $n$ large enough that $x_0^n$ belongs to $p^N\mathcal{G}_{[b,\alpha']} + \mathcal{G}_{[b,\infty[}^{>N}$ for $\alpha > \alpha' > a$, from which we conclude. $\square$

**Remark** The preceding lemma makes $\mathcal{G}_{[b,a)}$ into a complete valuation ring with the valuation given by

$$v_{[b,a)}(x) = \lim_{n\to\infty} \frac{k_n}{n}$$

where

$$k_n = \sup\{k \in \mathbb{N}, x^n \in \mathfrak{m}_{[b,a)}^k\}.$$

The following lemma provides a link between algebras $\mathcal{G}_{[b,a]}$ and Fontaine's rings.

**Lemma 2.6.**

1. *$\mathcal{G}_{[0,p]}$ injects continuously in $A_{crys}$.*
2. *Frobenius $\varphi$ of $A_{crys}$ and $\varphi_{\mathcal{G}}$ coincide on $\mathcal{G}_{[0,p]}$.*
3. *Any non zero element of $\mathcal{G}_{Y,[0,p]}$ is invertible in $\mathcal{G}_{Y,[p-1,p-1[} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.*
4. *The series defining $t/X$ converges in $\mathcal{G}_{[0,p]}$ where it is invertible.*

*Proof:* The first point consists in showing that $\frac{Y^{pen}}{p^n} \in A_{crys}$ for all $n$ and converges to 0. Let $E_\pi$ be an Eisenstein polynomial for $\pi$, it is of degree $e$ and $E_\pi(Y)$ generates $W^1(\tilde{\mathbf{E}}^+)$ so that $A_{crys}$ is the $p$-adic completion of $\tilde{\mathbf{A}}^+[\frac{E_\pi(Y)^n}{n!}]$ and it is obvious that $\frac{Y^{pen}}{p^n}$ belongs to this ring and $p$-adically converges to 0.
The second point is an immediate consequence of the first one.
Now let $x \in \mathcal{G}_{Y,[0,p]}$, then there exists a sequence $(a_n)_{n\in\mathbb{N}} \in \left(\tilde{\mathbf{A}}^+\left[\frac{1}{p}\right]\right)^{\mathbb{N}}$ such that

$$x = \sum_{n\in\mathbb{N}} a_n Y^n$$

with

$$\forall n \in \mathbb{N}\ ;\ epv_p(a_n) + n \geq 0.$$

Then,

$$\forall n \in \mathbb{N}\ ;\ e(p-1)v_p(a_n) + n \geq \frac{n}{p}$$

and for all $x$ non zero, $e(p-1)v_p(a_n)+n$ goes to $+\infty$ when $n \to +\infty$, it reaches its minimum $K$ a finite number of times and we fix $n_0$ the greater integer with $K = e(p-1)v_p(a_{n_0}) + n_0$, so that

$$e(p-1)v_p(a_n/a_{n_0}) + n - n_0 \geq 0 \quad \text{if } n \leq n_0 \tag{2.9}$$

$$e(p-1)v_p(a_n/a_{n_0}) + n - n_0 > 0 \quad \text{if } n > n_0 \tag{2.10}$$

and

$$\forall n > n_0\ ;\ e(p-1)v_p(a_n/a_{n_0}) + n - n_0 > \frac{n}{p} - K$$

hence it comes

$$\liminf_{n\to\infty} \frac{e(p-1)v_p(a_n/a_{n_0})+n-n_0}{n-n_0} \geq \frac{1}{p},$$

which, combined with (2.10), shows the existence of some $0<\lambda<1$ such that

$$e(p-1)v_p(a_n/a_{n_0})+n-n_0 \geq \lambda(n-n_0)$$

hence

$$e\frac{p-1}{1-\lambda}v_p(a_n/a_{n_0})+n-n_0 \geq 0.$$

This shows that for $a=\frac{p-1}{1-\lambda}>p-1$,

$$\sum_{n>n_0} \frac{a_n}{a_{n_0}}Y^{n-n_0} \in \mathfrak{m}_{[0,a]}.$$

Inequality (2.9) shows furthermore

$$\sum_{n=0}^{n_0-1} \frac{a_n}{a_{n_0}}Y^{n-n_0} \in \mathfrak{m}_{[p-1,\infty[}$$

and finally

$$\sum_{n\neq n_0} \frac{a_n}{a_{n_0}}Y^{n-n_0} \in \mathfrak{m}_{[p-1,a]}.$$

Then,

$$x=a_{n_0}Y^{n_0}(1+\epsilon)\ ;\ \epsilon\in\mathfrak{m}_{[p-1,a]}$$

is invertible in $\mathcal{G}_{Y,[p-1,a]}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \subset \mathcal{G}_{Y,[p-1,p-1[}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$.
Remark finally

$$X=[\varepsilon-1]+pv=Y^{ep/(p-1)}u+pv\ ;\ u,v\in\tilde{\mathbf{A}}^+$$

so that

$$X^{p-1}=Y^{ep}u'+pv'\ ;\ u',v'\in\tilde{\mathbf{A}}^+$$

from which we deduce that for $s$ prime to $p$,

$$\begin{aligned}
\frac{X^{p^r s-1}}{p^r s} &= \frac{X^{p^r(s-1)}}{s}\frac{X^{p^r-1}}{p^r} \\
&= X^{p^r(s-1)}\sum_{k=0}^{p^r-1}\frac{Y^{pek}}{p}^{\frac{p^r-1}{p-1}-k}p^r u_k \\
&= X^{p^r(s-1)}\sum_{k=0}^{p^r-1}\frac{Y^{pek}}{p}^{\frac{p^r-1}{p-1}-r}p^k u_k
\end{aligned}$$

where $u_k\in\tilde{\mathbf{A}}^+$. But

$$\frac{p^r-1}{p-1}\geq r$$

so that for all $n \geq 1$,

$$X^{n-1}/n \in \mathcal{G}_{[0,p]}$$

and $\frac{p^r-1}{p-1} - r$ goes to $+\infty$ with $r \to \infty$, which shows that $X^{n-1}/n$ converges $p$-adically to 0 in $\mathcal{G}_{[0,p]}$ and completes the proof of the lemma. $\square$

## 2.4 The Hilbert symbol of a formal group

### 2.4.1 The pairing associated with the Hilbert symbol

In this paragraph we express the Hilbert symbol of $F$ in terms of the Herr complex attached to $F[p^M]$.
Let us recall that the Hilbert symbol of a formal group is defined as the pairing:

$$\begin{array}{ccccl} K^* & \times & F(\mathfrak{m}_K) & \to & F[p^M] \\ (\alpha & , & \beta) & \mapsto & (\alpha, \beta)_{F,M} = r(\alpha)(\beta_1) -_F \beta_1 \end{array}$$

where $\beta_1 \in F(\mathfrak{m}_{\mathbb{C}_p})$ satisfies $p^M \mathrm{id}_F \beta_1 = \beta$ and $r : K^* \to G_K^{\mathrm{ab}}$ is the reciprocity map of local class field theory.
In fact, we will be interested in the pairing

$$\begin{array}{ccccl} F(\mathfrak{m}_K) & \times & G_K & \to & F[p^M] \\ (\beta & , & g) & \mapsto & (\beta, g]_{F,M} = g\beta_1 -_F \beta_1 \end{array}$$

where $\beta_1 \in F(m_{\mathbb{C}_p})$ satisfies $p^M \mathrm{id}_F \beta_1 = \beta$. Then

$$(\beta, r(\alpha)]_{F,M} = (\alpha, \beta)_{F,M}.$$

Put

$$\mathcal{R}(F) = \{(x_i)_{i \geq 0} \in F(\mathfrak{m}_{\mathbb{C}_p}) \text{ such that } x_0 \in F(\mathfrak{m}_K) \text{ and } (p\mathrm{id}_F)x_{i+1} = x_i \ \forall i \geq 0\}$$

then the Hilbert symbol is a mod $p$ reduction of the pairing

$$\begin{array}{ccccl} \mathcal{R}(F) & \times & G_K & \to & T(F) \\ (x & , & g) & \mapsto & (x, g]_{\mathcal{R}(F)} = (gx_i -_F x_i)_i \end{array}$$

with $((x, g]_{\mathcal{R}(F)})_M = (x_0, g]_{F,M}$ for any $x = (x_i) \in \mathcal{R}(F)$.
We can see this pairing as the connection map

$$F(\mathfrak{m}_K) \to H^1(K, T(F))$$

in the long exact sequence associated with the short exact one:

$$0 \to T(F) \to \varprojlim F(\mathfrak{m}_{\mathbb{C}_p}) \to F(\mathfrak{m}_{\mathbb{C}_p}) \to 0$$

where the transition maps in the inverse limit are $p\mathrm{id}_F$ and the last map is the projection on the first component. The ring $\mathcal{R}(F)$ is then the preimage of $F(\mathfrak{m}_K)$

by surjection $\varprojlim F(\mathfrak{m}_{\mathbb{C}_p}) \to F(\mathfrak{m}_{\mathbb{C}_p})$.
Let now $x \in F(\mathfrak{m}_{\tilde{\mathbf{E}}})$ be such that $\theta([x]) \in F(\mathfrak{m}_K)$. Then for all $g \in G_K$,

$$(g-1)\delta(x) \in F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}}))^{(\mathcal{A}-p)\circ l_{\mathcal{A}}=0} \simeq T(F)$$

where $\delta$ is the map defined at the end of Paragraph 2.2. The following diagram is commutative

$$\begin{tikzcd}
F(W(\mathfrak{m}_{\tilde{\mathbf{E}}}))_K^{(\mathcal{A}-p)\circ l_{\mathcal{A}}=0} \times G_K \arrow[r] & F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}}))^{(\mathcal{A}-p)\circ l_{\mathcal{A}}=0} \\
F(\mathfrak{m}_{\tilde{\mathbf{E}}})_K \times G_K \arrow[u, "\delta\times\mathrm{id}"] \arrow[d, "\iota\times\mathrm{id}"'] & \\
\mathcal{R}(F) \times G_K \arrow[r] & T(F) \arrow[uu, "j"']
\end{tikzcd}$$

where $\iota(x) = (\theta \circ \delta(p^{-s}\mathrm{id}_F(x)))_s = (\theta([p^{-s}\mathrm{id}_F(x))])_s$ and $F(\mathfrak{m}_{\tilde{\mathbf{E}}})_K$ (respectively $F(W(\mathfrak{m}_{\tilde{\mathbf{E}}}))_K$) stands for the set of $x \in F(\mathfrak{m}_{\tilde{\mathbf{E}}})$ (respectively $F(W(\mathfrak{m}_{\tilde{\mathbf{E}}}))$) with $\theta([x]) \in K$ (respectively $\theta(x) \in K$) and where the first pairing is simply

$$(u, g) \mapsto (g-1)u.$$

Fix now $\alpha \in F(\mathfrak{m}_K)$ and a lift $\xi$ of $\alpha$ in $F(\mathfrak{m}_{\tilde{\mathbf{E}}})$ which then satisfies

$$\theta([\xi]) = \alpha.$$

We get the equality

$$j((\iota(\xi), g]_{\mathcal{R}(F)}) = (g-1)\delta(\xi)$$

for all $g \in G_K$.
Choose now $\beta \in F(YW[[Y]])$ such that

$$\theta(\beta) = \alpha = \theta([\xi]).$$

Then for all $h \in G_L$,

$$(h-1)(\delta(\xi) -_F \beta) = j((\iota(\xi), h]_{\mathcal{R}(F)}).$$

Moreover, $\delta(\xi) -_F \beta \in F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}}))$ thus

$$l_{\mathcal{A}}(\delta(\xi) -_F \beta) \in (\mathrm{Fil}^1 A_{crys})^d$$

and

$$m_{\mathcal{A}}(\delta(\xi) -_F \beta) = \sum_{u \geq 1} F'_u \frac{\varphi^u(l_{\mathcal{A}}(\delta(\xi) -_F \beta))}{p}$$

converges in $A_{crys}^{h-n}$. Put now

$$\Lambda = \mathcal{V}^{-1} \begin{pmatrix} l_{\mathcal{A}}(\delta(\xi) -_F \beta) \\ m_{\mathcal{A}}(\delta(\xi) -_F \beta) \end{pmatrix} \in A_{crys}^h.$$

These are the coordinates of an element $\lambda$ in $D_{crys}(T(F)) \otimes A_{crys}$ in the basis $(o^1, \dots, o^h)$. And, for all $h \in G_L$,

$$(h-1)\lambda = (\iota(\xi), h]_{\mathcal{R}(F)} + \big((h-1)\mathcal{V}^{-1}\big) \begin{pmatrix} l_{\mathcal{A}}(\delta(\xi) -_F \beta) \\ m_{\mathcal{A}}(\delta(\xi) -_F \beta) \end{pmatrix}. \tag{2.11}$$

### 2.4.2 The approximated period matrix

Let us now explicitly compute the Hilbert symbol of $F$, *i.e.* the image of $\iota(\xi)$ in $H^1(K, F[p^M])$ which coincides with the one of $\alpha$. For that, we have to give a triple in the first homology group of the Herr complex of $F[p^M]$ corresponding to a cocycle representing the image of $\iota(\xi)$. Recall that if such a triple is written as $(x, y, z)$, then the associated cocycle is

$$g \mapsto (g-1)(-b) + \gamma^n \frac{\tau^m - 1}{\tau - 1} z + \frac{\gamma^n - 1}{\gamma - 1} y$$

where $g_{|\Gamma} = \gamma^n \tau^m$ and $b \in F[p^M] \otimes \tilde{\mathbf{A}}$ is a solution of

$$(\varphi - 1)b = x.$$

In particular, the image of $h \in G_L$ through this cocycle is $(h-1)(-b)$. Let us start with finding $b \in T(F) \otimes \tilde{\mathbf{A}}$ such that

$$\forall h \in G_L, \quad (h-1)b \equiv -(\iota(\xi), h]_{\mathcal{R}(F)} \mod p^M.$$

Equality (2.11) incites to build $b$ as an approximation of $-\lambda$. In fact, we will build $x$ by approximating $(\varphi - 1)(-\lambda)$, whose coordinates in the basis $(o^1, \dots, o^h)$ are

$$\mathcal{V}^{-1} \begin{pmatrix} (\frac{\mathcal{A}}{p} - 1) \circ l_{\mathcal{A}}(\beta) \\ 0 \end{pmatrix}.$$

Indeed, Lemma 2.2. shows that the action of Frobenius $\varphi$ is written in the basis $(o^1, \dots, o^h)\mathcal{V}^{-1}$:

$$\begin{pmatrix} \frac{\mathcal{A}}{p} & 0 \\ 0 & I_{h-d} \end{pmatrix}.$$

Because $(o^1 = (o_n^1)_n, \dots, o^h)$ is the fixed basis of $T(F)$, $(o_M^1, \dots, o_M^h)$ is a basis of $F[p^M]$ and we further fix $\hat{o}_M^1, \dots, \hat{o}_M^h$ elements in $F(YW[[Y]])$ such that for all $i$,

$$\theta(\hat{o}_M^i) = \hat{o}_M^i(\pi) = o_M^i.$$

Define then the matrix

$$\mathcal{V}_Y = \begin{pmatrix} p^M l_{\mathcal{A}}(\hat{o}_M^1) & \dots & p^M l_{\mathcal{A}}(\hat{o}_M^h) \\ p^M m_{\mathcal{A}}(\hat{o}_M^1) & \dots & p^M m_{\mathcal{A}}(\hat{o}_M^h) \end{pmatrix}$$

whose coefficients belong to $A_{crys}$, and more precisely to $W[[Y]]\left[\left[\frac{Y^{pe}}{p}\right]\right] = \mathcal{G}_{Y,[0,p]}$. From Lemma 2.6., $\mathcal{V}_Y$ is invertible in $\mathcal{G}_{Y,[p-1,p-1[} \otimes \mathbb{Q}_p$.

**Lemma 2.7.**

1. *$X\mathcal{V}_Y^{-1}$ has coefficients in $\mathcal{G}_{[0,p]} + \frac{p^M}{Y^{e/(p-1)}}\mathfrak{m}_{[\frac{1}{p-1},\infty[} \subset \mathcal{G}_{[\frac{1}{p-1},p]}$ and thus*

$$\varphi_{\mathcal{G}}(X\mathcal{V}_Y^{-1}) \in \mathcal{G}_{[p/(p-1),p]}.$$

2. *The matrix $\mathcal{V}_Y^{-1}$ has coefficients in $\frac{1}{Y^{ep/(p-1)}}\mathcal{G}_{[1,p]}$, then in $\frac{1}{Y^{\lceil ep/(p-1)\rceil}}\mathcal{G}_{Y,[1,p]}$ and*

$$\mathcal{V}_Y^{-1} \equiv \mathcal{V}^{-1} \mod \frac{p^M}{Y^{e(p+1)/(p-1)}}\mathfrak{m}_{[1,p]}.$$

3. *The principal part $\mathcal{V}_Y^{(-1)}$ of $\mathcal{V}_Y^{-1}$ has $p$-entire coefficients and its derivative $\frac{d}{dY}\mathcal{V}_Y^{(-1)}$ has coefficients in $p^M\tilde{\mathbf{A}}$.*

4. *The matrix $X\mathcal{V}_Y^{(-1)}$ has coefficients in $\tilde{\mathbf{A}}^+ + p^M\tilde{\mathbf{A}}$.*

*Proof:* We use the strategy of Paragraph 3.4. in [Abr97]. Let us recall that Abrashkin there showed

$$\begin{aligned} p^M l_{\mathcal{A}}(\hat{o}_M^i) &\in \left(E_\pi(Y)YW[[Y]] + \frac{E_\pi(Y)^p}{p}W[[Y]]\left[\left[\frac{Y^{ep}}{p}\right]\right]\right)^n \\ p^M m_{\mathcal{A}}(\hat{o}_M^i) &\in \left(YW[[Y]] + \frac{Y^{ep}}{p}W[[Y]]\left[\left[\frac{Y^{ep}}{p}\right]\right]\right)^{h-n} \end{aligned}$$

and

$$\begin{aligned} \mathbf{l}(o^i) - p^M l_{\mathcal{A}}(\hat{o}_M^i) &\in p^M\left(E_\pi(Y)W(\mathfrak{m}_{\tilde{\mathbf{E}}}) + \frac{E_\pi(Y)^p}{p}\tilde{\mathbf{A}}^+\left[\left[\frac{Y^{ep}}{p}\right]\right]\right)^n \\ \mathbf{m}(o^i) - p^M m_{\mathcal{A}}(\hat{o}_M^i) &\in p^M\left(W(\mathfrak{m}_{\tilde{\mathbf{E}}}) + \frac{Y^{ep}}{p}\tilde{\mathbf{A}}^+\left[\left[\frac{Y^{ep}}{p}\right]\right]\right)^{h-n}. \end{aligned}$$

Let $\mathcal{V}^D$ be the matrix of the group dual to $F$. It satisfies the relation:

$${}^t\mathcal{V}^D\ \mathcal{V} = tI_h.$$

And one can then write

$${}^t\mathcal{V}^D\ \mathcal{V}_Y \equiv tI_h \mod p^M\left(E_\pi(Y)W(\mathfrak{m}_{\tilde{\mathbf{E}}}) + \frac{E_\pi(Y)^p}{p}\tilde{\mathbf{A}}^+\left[\left[\frac{Y^{ep}}{p}\right]\right]\right)$$

in particular,

$${}^t\mathcal{V}^D\ \mathcal{V}_Y \equiv tI_h \mod p^M\left(Y^eW(\mathfrak{m}_{\tilde{\mathbf{E}}}) + \frac{Y^{ep}}{p}\tilde{\mathbf{A}}^+\left[\left[\frac{Y^{ep}}{p}\right]\right]\right).$$

Remark, because of Lemma 2.6., that the element $t/X$ converges in $\mathcal{G}^*_{[0,p]}$, and

$$X = \omega[\varepsilon^{1/p}-1] = E_\pi(Y)Y^{e/(p-1)}v\ ;\ v \in \mathcal{G}^*_{[\frac{1}{p-1},\infty[},$$

so that

$${}^t\mathcal{V}^D\ \mathcal{V}_Y = t(I_h - p^M u)$$

with

$$\begin{aligned} u \in \frac{E_\pi(Y)}{t}W(\mathfrak{m}_{\tilde{\mathbf{E}}}) + \frac{Y^{ep}}{pt}\mathcal{G}_{[0,p]} \quad &\subset \quad \frac{1}{Y^{e/(p-1)}}\mathfrak{m}_{[\frac{1}{p-1},p]} + \frac{Y^{e\frac{p^2-2p}{p-1}}}{p}\mathcal{G}_{[\frac{1}{p-1},p]} \\ &\subset \quad \frac{1}{Y^{e/(p-1)}}\mathfrak{m}_{[\frac{1}{p-1},p]} \subset \frac{1}{p}\mathfrak{m}_{[\frac{1}{p-1},p]} \end{aligned}$$

thus

$$\mathcal{V}_Y^{-1} = \frac{1}{t}\left(\sum_{n\in\mathbb{N}} p^{Mn}u^n\right){}^t\mathcal{V}^D \in \frac{1}{t}\mathcal{G}_{[\frac{1}{p-1},p]}$$

and then we deduce the first point ; and even

$$\mathcal{V}_Y^{-1} \equiv \mathcal{V}^{-1} \mod \frac{p^M}{tY^{e/(p-1)}}\mathfrak{m}_{[\frac{1}{p-1},p]}$$

or

$$\mathcal{V}_Y^{-1} \in \frac{1}{t}\mathcal{G}_{[0,p]} + \frac{p^M}{Y^{e/(p-1)}}\mathfrak{m}_{[\frac{1}{p-1},p]}.$$

Recall

$$t = E_\pi(Y)\varphi^{-1}(X)u'\ ;\ u' \in \mathcal{G}^*_{[0,p]}$$

and remark that because $E_\pi$ is an Eisenstein polynomial, $E_\pi(Y)$ and $Y^e$ are associated in $\mathcal{G}_{[1,\infty[}$ ; finally, with the above computation, we deduce that $t$ and $Y^{ep/(p-1)}$ are associated in $\mathcal{G}_{[1,p]}$. Then

$$\mathcal{V}_Y^{-1} \in \frac{1}{Y^{ep/(p-1)}}\mathcal{G}_{[1,p]} + \frac{p^M}{Y^{e(p+1)/(p-1)}}\mathfrak{m}_{[1,p]} \subset \frac{1}{Y^{ep/(p-1)}}\mathcal{G}_{[1,p]}.$$

So, $Y^{\lceil ep/(p-1)\rceil}\mathcal{V}_Y^{-1}$ has coefficients in $\mathcal{G}_{Y,[p-1,p-1[}\left[\frac{1}{p}\right]\bigcap\mathcal{G}_{[1,p]} = \mathcal{G}_{Y,[1,p]}$ because of Lemma 2.4..
Let us further deduce that $\mathcal{V}_Y^{(-1)}$ has $p$-entire coefficients. It is to show that any element

$$x = \sum_{n\in\mathbb{Z}} a_nY^n \in \frac{1}{Y^{\lceil ep/(p-1)\rceil}}\mathcal{G}_{Y,[1,p]}$$

satisfies $a_n \in W$ for all $n \leq 0$. But that means that

$$Y^{\lceil ep/(p-1)\rceil} x = \sum_{n\in\mathbb{Z}} a_n Y^{n+\lceil ep/(p-1)\rceil} \in \mathcal{G}_{Y,[1,p]}$$

and thus if $v_p(a_n) \leq 1$, the following inequality holds

$$n + ep/(p-1) \geq ep$$

hence

$$n \geq \frac{(p-2)ep}{p-1} > 0$$

and $\mathcal{V}_Y^{(-1)}$ has $p$-entire coefficients.
For the third point, let us recall the argument of Lemma 4.5.4 in [Abr97]. Write

$$\frac{d}{dY}\mathcal{V}_Y^{(-1)} = -\mathcal{V}_Y^{(-1)}\left(\frac{d}{dY}\mathcal{V}_Y\right)\mathcal{V}_Y^{(-1)}$$

and because differentials of $l_{\mathcal{A}}$ and $m_{\mathcal{A}}$ have coefficients in $W$, one gets

$$\frac{d}{dY}\mathcal{V}_Y \in p^M M_h(W[[Y]])$$

so that

$$\frac{d}{dY}\mathcal{V}_Y^{(-1)} \in \mathcal{G}_{Y,[p-1,p-1[} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \bigcap \frac{1}{Y^{2ep/(p-1)}}\mathcal{G}_{[1,p]}$$

and the same argument as above permits to conclude (we get then the inequality $n \geq \frac{(p-3)ep}{p-1} \geq 0$).
Finally, the proof of Point 4. is the same as the one of Proposition 3.7, Point *d)* in [Abr97]. Let us write it in the following way: we know on the one hand that $\mathcal{V}_Y^{(-1)}$ and then also $X\mathcal{V}_Y^{(-1)}$ has $p$-entire coefficients, so they have coefficients in $\mathcal{G}_{[p-1,\infty[}\left[\frac{1}{Y}\right]$ and that $\mathcal{U} = X(\mathcal{V}_Y^{-1} - \mathcal{V}_Y^{(-1)})$ has coefficients in $\mathcal{G}_{[0,p-1[}\left[\frac{1}{p}\right]$. On the other hand, Lemma 2.7. tells

$$X\mathcal{V}_Y^{-1} \in M_h\left(\mathcal{G}_{[0,p]} + p^{M-1}\mathcal{G}_{[1/(p-1),\infty[}\right).$$

Remark

$$\mathcal{G}_{[1/(p-1),\infty[} = \tilde{\mathbf{A}}^+\left[\left[\frac{p}{Y^{e/(p-1)}}\right]\right] = \tilde{\mathbf{A}}^+ + \frac{p}{Y^{e/(p-1)}}\mathcal{G}_{[1/(p-1),\infty[}$$

Thus we can write

$$X\mathcal{V}_Y^{-1} = M_1 + p^M M_2$$

with $M_1$ having coefficients in $\mathcal{G}_{[0,p]}$ and $M_2$ in $\frac{1}{Y^{e/(p-1)}}\mathcal{G}_{[1/(p-1),\infty[} \subset \tilde{\mathbf{A}}$. So

$$X\mathcal{V}_Y^{(-1)} - p^M M_2 = M_1 - \mathcal{U}$$

has coefficients in $\mathcal{G}_{[p-1,\infty[}\left[\frac{1}{Y}\right] \bigcap \mathcal{G}_{[0,p-1[}\left[\frac{1}{p}\right] = \tilde{\mathbf{A}}^+$, as desired. $\square$

Remark that if $x \in F(W(\mathfrak{m}_{\tilde{\mathbf{E}}}))$ it can be written as

$$x = [x_0] +_F u$$

with $u \in F(pW(\mathfrak{m}_{\tilde{\mathbf{E}}}))$, and thus

$$\begin{aligned}\left(\frac{\mathcal{A}}{p} - 1\right) \circ l_{\mathcal{A}}(x) &= \left(\frac{\mathcal{A}}{p} - 1\right) \circ l_{\mathcal{A}}([x_0]) + \left(\frac{\mathcal{A}}{p} - 1\right) \circ l_{\mathcal{A}}(u) \\ &= [x_0] + \left(\frac{\mathcal{A}}{p} - 1\right) \circ l_{\mathcal{A}}(u) \in W(\mathfrak{m}_{\tilde{\mathbf{E}}})^d\end{aligned}$$

since $l_{\mathcal{A}}(u) \in pW(\mathfrak{m}_{\tilde{\mathbf{E}}})^d$. In particular

$$\left(\frac{\mathcal{A}}{p} - 1\right) \circ l_{\mathcal{A}}(\beta) \in W(\mathfrak{m}_{\tilde{\mathbf{E}}})^d,$$

so that

$$\mathcal{V}_Y^{(-1)} \begin{pmatrix} \left(\frac{\mathcal{A}}{p} - 1\right) \circ l_{\mathcal{A}}(\beta) \\ 0 \end{pmatrix} \in \tilde{\mathbf{A}}^h.$$

### 2.4.3 An explicit computation of the Hilbert symbol

We come now to the proposition that explicitly gives the desired triple. The basic ingredient is Proposition 3.8 of [Abr97] which provides the $x$ coordinate of the triple and allows us to prove that $y$ is zero. However, in order to get $z$, we have to carry Abrashkin's computations to the higher order. Indeed, we already know that $z$ belongs to $W(\mathfrak{m}_{\tilde{\mathbf{E}}})$, but we need to specify its value modulo $XW(\mathfrak{m}_{\tilde{\mathbf{E}}})$.
Let us recall the results we are going to use

**Proposition 2.2.**
*Let $U$ be the principal part of* $\mathcal{V}_Y^{(-1)} \begin{pmatrix} \left(\frac{\mathcal{A}}{p} - 1\right) \circ l_{\mathcal{A}}(\beta) \\ 0 \end{pmatrix}$ *and* $\hat{x} = (o^1, \ldots, o^h)U$. *Then*

1. $U \in (W[[\frac{1}{Y}]] \cap \tilde{\mathbf{A}})^h$
2. *Let* $\hat{b} \in T(F) \otimes \tilde{\mathbf{A}}$ *be a solution of* $(\varphi - 1)\hat{b} = \hat{x}$ *then for any* $g \in G_K$,

$$(g-1)\hat{b} \equiv (\beta(\pi), g]_{F,M} \mod p^M \tilde{\mathbf{A}} + W(\mathfrak{m}_{\tilde{\mathbf{E}}}).$$

*Proof:*
The first point can be shown like Point 3. of Lemma 2.7. above.
The second point can be viewed as a reformulation of Proposition 3.8 of [Abr97]. Let us give another proof.
Let us recall from Lemma 2.7. that

$$\mathcal{V}_Y^{-1} \equiv \mathcal{V}^{-1} \mod \frac{p^M}{Y^{e(p+1)/(p-1)}} \mathfrak{m}_{[1,p]}$$

so that there is $\delta \in \frac{p^M}{Y^{e(p+1)/(p-1)}}\mathfrak{m}_{[1,p]}$ such that

$$X\mathcal{V}_Y^{-1} = X\mathcal{V}^{-1} + X\delta.$$

Write $\delta = \delta_1 + \delta_2$ with $\delta_1 \in p^{M-1}Y^{e(p^2-2p-1)/(p-1)}\mathcal{G}_{[0,p]}$ and $\delta_2 \in \frac{p^M}{Y^{e(p+1)/(p-1)}}\mathfrak{m}_{[1,\infty[}$. Let us recall that we write $\mathcal{V}_Y^{-1} = \mathcal{V}_Y^{(-1)} + \mathcal{U}$ so that

$$X\mathcal{V}_Y^{(-1)} - \delta_2 = X\mathcal{V}^{-1} + X\delta_1 - X\mathcal{U}$$

has coefficients in $\mathcal{G}_{[p-1,\infty[}\left[\frac{1}{Y}\right] \bigcap \mathcal{G}_{[0,p-1[}\left[\frac{1}{p}\right] = \tilde{\mathbf{A}}^+$.
Then, if $\mathcal{B}$ is a matrix with coefficients in $\tilde{\mathbf{A}}$ such that

$$(\varphi - 1)\mathcal{B} = \left(\mathcal{V}_Y^{(-1)} - \delta_2\right)\begin{pmatrix}\left(\frac{\mathcal{A}}{p} - 1\right) \circ l_{\mathcal{A}}(\beta) \\ 0\end{pmatrix}, \tag{2.12}$$

write as in Paragraph 1.7,

$$(\varphi - \omega)(X_1\mathcal{B}) = \left(X\mathcal{V}_Y^{(-1)} - X\delta_2\right)\begin{pmatrix}\left(\frac{\mathcal{A}}{p} - 1\right) \circ l_{\mathcal{A}}(\beta) \\ 0\end{pmatrix}$$

has coefficients in $\tilde{\mathbf{A}}^+$ so that, by successive approximations modulo $p^k$ and since $\tilde{\mathbf{E}}^+$ is integrally closed, we get

$$\mathcal{B} \in \frac{1}{X_1}\tilde{\mathbf{A}}^+ \subset \mathrm{Fil}^0 B_{crys}.$$

Still write

$$\Lambda = \mathcal{V}^{-1}\begin{pmatrix} l_{\mathcal{A}}(\delta(\xi) -_F \beta) \\ m_{\mathcal{A}}(\delta(\xi) -_F \beta)\end{pmatrix} \in \left(\mathrm{Fil}^0 A_{crys}\right)^h.$$

We compute

$$(\varphi - 1)\left(\mathcal{B} - \Lambda\right) = (\delta_1 - \mathcal{U})\begin{pmatrix}\left(\frac{\mathcal{A}}{p} - 1\right) \circ l_{\mathcal{A}}(\beta) \\ 0\end{pmatrix}$$

Since $\delta_1' = \delta_1\begin{pmatrix}\left(\frac{\mathcal{A}}{p} - 1\right) \circ l_{\mathcal{A}}(\beta) \\ 0\end{pmatrix}$ has coefficients in $Y\mathcal{G}_{[0,p]}$, the series $-\sum_{n\in\mathbb{N}} \varphi^n(\delta_1')$ converges to an element $\Delta_1 \in Y\mathcal{G}_{[0,p]}$ satisfying

$$(\varphi - 1)(\Delta_1) = \delta_1'.$$

Likewise $\delta_2' = \mathcal{U}\begin{pmatrix}\left(\frac{\mathcal{A}}{p} - 1\right) \circ l_{\mathcal{A}}(\beta) \\ 0\end{pmatrix}$ has coefficients in $YW[[Y]] + \frac{Y^{ep-\lceil ep/(p-1)\rceil}}{p}\mathcal{G}_{Y,[0,p]}$ so that the series $-\sum_{n\in\mathbb{N}} \varphi^n(\delta_2')$ converges to an element $\Delta_2$ with coefficients in $YW[[Y]] + \frac{Y^{ep-\lceil ep/(p-1)\rceil}}{p}\mathcal{G}_{Y,[0,p]}$ satisfying

$$(\varphi - 1)(\Delta_2) = \delta_2'.$$

Finally,

$$(\varphi - 1)\left(\mathcal{B} - \Lambda - \Delta_1 + \Delta_2\right) = 0$$

with $\mathcal{B} - \Lambda - \Delta_1 + \Delta_2$ having coefficients in $\mathrm{Fil}^0 B_{crys}$. And the fact that

$$(\mathrm{Fil}^0 B_{crys})_{\varphi=1} = \mathbb{Q}_p$$

shows

$$\mathcal{B} - \Lambda - \Delta_1 + \Delta_2 \in \mathbb{Q}_p.$$

Then, for $g \in G_K$, $(g-1)\left(\mathcal{B} - \Lambda - \Delta_1 + \Delta_2\right) = 0$ so that

$$\begin{aligned}
(g-1)(\mathcal{B}) &= (g-1)(\Lambda + \Delta_1 - \Delta_2) \\
(g-1)(\mathcal{B}) &= (g-1)\left(\mathcal{V}^{-1}\begin{pmatrix} l_{\mathcal{A}}(\delta(\xi) -_F \beta) \\ m_{\mathcal{A}}(\delta(\xi) -_F \beta)\end{pmatrix}\right) + (g-1)(\Delta_1 - \Delta_2) \\
(g-1)(\mathcal{B}) &= \left((g-1)\mathcal{V}^{-1}\right) g \begin{pmatrix} l_{\mathcal{A}}(\delta(\xi) -_F \beta) \\ m_{\mathcal{A}}(\delta(\xi) -_F \beta)\end{pmatrix} \\
&\quad + \mathcal{V}^{-1}(g-1)\left(\begin{pmatrix} l_{\mathcal{A}}(\delta(\xi) -_F \beta) \\ m_{\mathcal{A}}(\delta(\xi) -_F \beta)\end{pmatrix}\right) + (g-1)(\Delta_1 - \Delta_2)
\end{aligned}$$

Now, we remark that

$$\begin{aligned}
\mathbf{l}(o^i) - g\mathbf{l}(o^i) &\in p^M\left(E_\pi(Y)W(\mathfrak{m}_{\tilde{\mathbf{E}}}) + \frac{E_\pi(Y)^p}{p}\tilde{\mathbf{A}}^+\left[\left[\frac{Y^{ep}}{p}\right]\right]\right)^n \\
\mathbf{m}(o^i) - g\mathbf{m}(o^i) &\in p^M\left(W(\mathfrak{m}_{\tilde{\mathbf{E}}}) + \frac{Y^{ep}}{p}\tilde{\mathbf{A}}^+\left[\left[\frac{Y^{ep}}{p}\right]\right]\right)^{h-n}
\end{aligned}$$

so that $g\mathcal{V}$ enjoys the same approximation properties as $\mathcal{V}_Y$, hence $(g-1)\mathcal{V}^{-1}$ has coefficients in $\frac{p^M}{Y^{e(p+1)/(p-1)}}\mathfrak{m}_{[1,p]}$. Thus coefficients of $\left((g-1)\mathcal{V}^{-1}\right) g \begin{pmatrix} l_{\mathcal{A}}(\delta(\xi) -_F \beta) \\ m_{\mathcal{A}}(\delta(\xi) -_F \beta)\end{pmatrix} + (g-1)(\Delta_1 - \Delta_2)$ lie in $\frac{p^M}{Y^{e(p+1)/(p-1)}}\mathfrak{m}_{[1,\infty[} + \frac{Y}{p}\mathcal{G}_{[0,p]}$. Let us recall finally that

$$\mathcal{V}^{-1}(g-1)\left(\begin{pmatrix} l_{\mathcal{A}}(\delta(\xi) -_F \beta) \\ m_{\mathcal{A}}(\delta(\xi) -_F \beta)\end{pmatrix}\right)$$

are the coordinates of $(\iota(\xi), g]_{\mathcal{R}(F)}$ in the basis $(o_1, \ldots, o^h)$ of $T(F)$, and thus it is congruent to the coordinates of $(\beta(\pi), g]_{F,M}$ modulo $p^M\mathbb{Z}_p$. We get

$$(g-1)(\mathcal{B}) - (\iota(\xi), g]_{\mathcal{R}(F)} \in \frac{p^M}{Y^{e(p+1)/(p-1)}}\mathfrak{m}_{[1,\infty[} + \frac{Y}{p}\mathcal{G}_{[0,p]}.$$

Recall now that coefficients of $\mathcal{B}$ lie in

$$\frac{1}{X_1}\tilde{\mathbf{A}}^+ \subset \frac{1}{Y^{e/(p-1)}}\mathcal{G}_{[\frac{1}{p-1},\infty[}.$$

Gathering all information, we deduce the existence of $u_1 \in \frac{1}{Y^{e(p+1)/(p-1)}}\mathfrak{m}_{[1,\infty[}$ and $u_2 \in \frac{Y}{p}\mathcal{G}_{[0,p]}$ such that

$$(g-1)(\mathcal{B}) - (\iota(\xi), g]_{\mathcal{R}(F)} - p^M u_1 = u_2$$

has coefficients in

$$\frac{1}{Y^{e(p+1)/(p-1)}}\mathcal{G}_{[\frac{1}{p-1},\infty[} \bigcap \frac{Y}{p}\mathcal{G}_{[0,p]} = Y\tilde{\mathbf{A}}^+ \subset W(\mathfrak{m}_{\tilde{\mathbf{E}}})$$

so that $(g-1)(\mathcal{B})$ is congruent to coordinates of $(\beta(\pi), g]_{F,M}$ modulo $p^M\tilde{\mathbf{A}} + W(\mathfrak{m}_{\tilde{\mathbf{E}}})$.
To finish the proof, just recall Equality (2.12): there is $\delta_2 \in \frac{p^M}{Y^{e(p+1)/(p-1)}}\mathfrak{m}_{[1,\infty[} \subset p^M\tilde{\mathbf{A}}$ such that

$$(\varphi-1)\mathcal{B} = \left(\mathcal{V}_Y^{(-1)} - \delta_2\right)\begin{pmatrix}\left(\frac{\mathcal{A}}{p}-1\right)\circ l_{\mathcal{A}}(\beta) \\ 0\end{pmatrix}$$

And surjectivity of $\varphi - 1$ on $\tilde{\mathbf{A}}$ permits to conclude. $\square$

**Remark** It is possible to get rid of $A_{crys}$ in the proof by studying the action of $(\varphi-1)$ on $\mathcal{G}_{[0,p]}\left[\frac{1}{Y}\right]$.
We will use this result in the following specified form.

**Proposition 2.3.**
*Let $\beta \in F(YW[[Y]])$ and $\alpha = \theta(\beta) = \beta(\pi) \in F(\mathfrak{m}_K)$. Put*

$$x = (o^1, \ldots, o^h)\mathcal{V}_Y^{(-1)}\begin{pmatrix}(\frac{\mathcal{A}}{p}-1)\circ l_{\mathcal{A}}(\beta) \\ 0\end{pmatrix} \in \tilde{D}_L(T(F))$$

*then there exists*

$$z \in \tilde{D}_L(T(F)) \cap T(F)\otimes W(\mathfrak{m}_{\tilde{\mathbf{E}}})$$

*unique modulo $p^M$ such that the class of the triple $(x, 0, z)$ corresponds to the image of $\alpha$ by the Kummer map $F(\mathfrak{m}_K) \to H^1(K, F[p^M])$.*
*Moreover, $z$ is congruent to*

$$XY\mathcal{V}_Y^{(-1)}\frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta)\end{pmatrix} \quad \text{mod } XW(\mathfrak{m}_{\tilde{\mathbf{E}}}).$$

### 2.4.4 Proof of Proposition 2.3.

We use Proposition 2.2., and remark that

$$\hat{x} - x \in T(F)\otimes YW[[Y]] \subset (\varphi-1)(T(F)\otimes YW[[Y]]).$$

So, if $b \in T(F)\otimes\tilde{\mathbf{A}}$ satisfies $(\varphi-1)b = x$, then for any $g \in G_K$,

$$(g-1)b \equiv (\alpha, g]_{F,M} \quad \text{mod } p^M\tilde{\mathbf{A}} + W(\mathfrak{m}_{\tilde{\mathbf{E}}}).$$

Thus for any $h \in G_L$,

$$(h-1)b \equiv (\alpha, h]_{F,M} \mod p^M T(F)$$

for $(h-1)b \in \ker(\varphi - 1) = T(F)$.
We deduce that there exist $y, z \in \tilde{D}_L(T(F))$ unique modulo $p^M$ such that the class of the triple $(x, y, z)$ corresponds to the image of $\alpha$ in $H^1(K, F[p^M])$ ; indeed let $(x_1, y_1, z_1)$ be such a triple, and $b_1 \in T(F) \otimes \tilde{\mathbf{A}}$ a solution of $(\varphi - 1)b_1 = x_1$ then for all $h \in G_L$,

$$(h-1)(b_1 - b) \equiv 0 \mod p^M,$$

$$\text{thus, } b_1 - b \in \tilde{D}_L(F[p^M]),$$

which shows that the class of

$$(x, y_1 + (\gamma - 1)(b - b_1), z_1 + (\tau - 1)(b - b_1))$$

corresponds to the same class as $(x_1, y_1, z_1)$ and, if $x$ is fixed, this triple is unique.
Let us now determine $y$: let $\tilde{\gamma}$ lift $\gamma$ then

$$(\tilde{\gamma} - 1)(-b) + y = (\alpha, \tilde{\gamma}]_{F,M} \equiv (\tilde{\gamma} - 1)(-b) \mod p^M \tilde{\mathbf{A}} + W(\mathfrak{m}_{\tilde{\mathbf{E}}})$$

hence, for $(\tilde{\gamma} - 1)(-b) \in T(F)$,

$$y \in T(F) \otimes W(\mathfrak{m}_{\tilde{\mathbf{E}}}) \cap T(F) = \{0\}.$$

Likewise, let $\tilde{\tau}$ lift $\tau$ then

$$(\tilde{\tau} - 1)(-b) + z = (\alpha, \tilde{\tau}]_{F,M} \equiv (\tilde{\tau} - 1)(-b) \mod p^M \tilde{\mathbf{A}} + W(\mathfrak{m}_{\tilde{\mathbf{E}}})$$

hence $z \in T(F) \otimes W(\mathfrak{m}_{\tilde{\mathbf{E}}})$.
Thus $z$ belongs to $T(F) \otimes W(\mathfrak{m}_{\tilde{\mathbf{E}}})$ and satisfies

$$(\tau - 1)x = (\varphi - 1)z.$$

This uniquely determines $z$ since $\varphi - 1$ is injective on $T(F) \otimes W(\mathfrak{m}_{\tilde{\mathbf{E}}})$. In order to specify $z$, we need the following lemma:

**Lemma 2.8.**

1. *For all $U \in W[[Y]]$, the following congruence holds*

$$(\tau - 1)\mathcal{V}_Y^{(-1)} U \equiv XY\mathcal{V}_Y^{(-1)} \frac{dU}{dY} \mod XW(\mathfrak{m}_{\tilde{\mathbf{E}}}) + p^M \tilde{\mathbf{A}}.$$

2. *There exists $u \in \mathfrak{m}_{[p/(p-1),p]}$ such that*

$$\varphi_{\mathcal{G}}(X\mathcal{V}_Y^{-1}) = (\varphi_{\mathcal{G}}(X)\mathcal{V}_Y^{-1}\mathcal{E}^{-1} + p^M u) \begin{pmatrix} \frac{1}{p} I_d & 0 \\ 0 & I_{h-d} \end{pmatrix}$$

*Proof of the lemma:* For Point 2., we first specify $(\tau-1)\mathcal{V}_Y^{(-1)}$. Remark that if $f(Y)$ is a series in $W\{\{Y\}\} \cap \tilde{\mathbf{A}}$,

$$(\tau-1)f(Y) = \sum_{n\geq 1} \frac{(XY)^n}{n!} f^{(n)}(Y)$$

Thus for $\mathcal{V}_Y^{(-1)}$:

$$(\tau-1)\mathcal{V}_Y^{(-1)} = XY\frac{d}{dY}\mathcal{V}_Y^{(-1)} + \frac{(XY)^2}{2}\frac{d^2}{dY^2}\mathcal{V}_Y^{(-1)} + \sum_{n\geq 3}\frac{(XY)^n}{n!}\frac{d^n}{dY^n}\mathcal{V}_Y^{(-1)}$$

We then have to estimate $\frac{(XY)^n}{n!}\frac{d^n}{dY^n}\mathcal{V}_Y^{(-1)}$. Lemma 2.7. shows

$$\frac{d}{dY}\mathcal{V}_Y^{-1} = p^M \mathcal{V}_Y^{-1}\widetilde{W}\mathcal{V}_Y^{-1},$$

where $\widetilde{W} \in W[[Y]]$ and the principal part of $\mathcal{V}_Y^{-1}\widetilde{W}\mathcal{V}_Y^{-1}$ is entire. Thus, on the one hand

$$XY\frac{d}{dY}\mathcal{V}_Y^{(-1)} + \frac{(XY)^2}{2}\frac{d^2}{dY^2}\mathcal{V}_Y^{(-1)} \in p^M\tilde{\mathbf{A}}$$

and on the other hand one can write

$$\frac{d^n}{dY^n}\mathcal{V}_Y^{-1} = \sum_{k=1}^{n} p^{Mk} w_{n,k}$$

where the $w_{n,k}$ are sums of terms of the form

$$\mathcal{V}_Y^{-1}\widetilde{W}_{n,1}\mathcal{V}_Y^{-1}\widetilde{W}_{n,2}\ldots\widetilde{W}_{n,k}\mathcal{V}_Y^{-1},$$

where the $\widetilde{W}_{n,i} \in W[[Y]]$ are derivatives of $\widetilde{W}$.
Recall that $\mathcal{V}_Y^{-1}$ has coefficients in $\frac{1}{X}\left(\mathcal{G}_{[0,p]} + p^{M-1}\mathfrak{m}_{[1/(p-1),p]}\right)$, then

$$\mathcal{V}_Y^{-1}\widetilde{W}_{n,1}\mathcal{V}_Y^{-1}\widetilde{W}_{n,2}\ldots\widetilde{W}_{n,k}\mathcal{V}_Y^{-1} \in M_h\left(\frac{1}{X^{k+1}}\mathcal{G}_{[0,p]} + p^{M-1}\left(\frac{1}{X^{k+1}}\mathfrak{m}_{[1/(p-1),p]}\right)\right).$$

Suppose $1 < k < n-1$. Since

$$v_p(n!) \leq \lfloor n/(p-1)\rfloor = n',$$

there exists $u \in \mathbb{Z}_p$ such that

$$p^{Mk}\frac{(XY)^n}{n!} = Y^n X^{k+2} u \frac{X^{n-k-2}}{p^{n'-Mk}}.$$

Since $p > 2$ and $k > 1$,

$$(n'-Mk)(p-1) \leq n-k-2 \text{ and } \frac{X^{n-k-2}}{p^{n'-Mk}} \in W\left[\left[\frac{X^{p-1}}{p}\right]\right] \subset \mathcal{G}_{[0,p]}.$$

Thus,

$$p^{Mk}\frac{(XY)^{n-1}}{n!}w_{n,k} \in \mathfrak{m}_{[0,p]} + p^{M-1}\mathfrak{m}_{[1/(p-1),p]}.$$

Let now $k = 1$, write

$$w_{n,1} = \mathcal{V}_Y^{-1}\frac{d^{n-1}}{dY^{n-1}}\widetilde{W}\mathcal{V}_Y^{-1} = (n-1)!\mathcal{V}_Y^{-1}\widetilde{W}'\mathcal{V}_Y^{-1}$$

and

$$\frac{(XY)^{n-1}}{n!}p^M w_{n,1} = \frac{(XY)^{n-1}}{n}\mathcal{V}_Y^{-1}\widetilde{W}'\mathcal{V}_Y^{-1} = \frac{X^{n-p}}{n}Y^n X^{p-1}\mathcal{V}_Y^{-1}\widetilde{W}'\mathcal{V}_Y^{-1}$$

has coefficients in $\mathfrak{m}_{[0,p]} + p^{M-1}\mathfrak{m}_{[1/(p-1),p]}$ as before.
Let $k = n$, one has obviously

$$w_{n,n} = n!\mathcal{V}_Y^{-1}\widetilde{W}_{n,1}\mathcal{V}_Y^{-1}\widetilde{W}_{n,2}\dots\widetilde{W}_{n,n}\mathcal{V}_Y^{-1}$$

where for all $1 \leq i \leq n$, $\widetilde{W}_{n,i} = \widetilde{W}$, so that

$$\frac{(XY)^n}{n!}p^{Mn}w_{n,n} \in p^{Mn}\frac{1}{X}\left(\mathcal{G}_{[0,p]} + p^{M-1}\mathfrak{m}_{[1/(p-1),p]}\right).$$

Finally, for $k = n-1$, since $v_p(n!) \leq n/(p-1) \leq n-1$,

$$\frac{(XY)^n}{n!}p^{M(n-1)}w_{n,n-1} \in Y^n\left(\mathcal{G}_{[0,p]} + p^{M-1}\mathcal{G}_{[1/(p-1),p]}\right).$$

The same argument as 1. then shows that for all $n > 2$,

$$\frac{(XY)^n}{n!}\frac{d^n}{dY^n}\mathcal{V}_Y^{(-1)} \in XW(\mathfrak{m}_{\tilde{\mathbf{E}}}) + p^M\tilde{\mathbf{A}}$$

hence

$$(\tau-1)\mathcal{V}_Y^{(-1)} \in XW(\mathfrak{m}_{\tilde{\mathbf{E}}}) + p^M\tilde{\mathbf{A}}.$$

Point 2. then follows from the equality

$$(\tau-1)\mathcal{V}_Y^{(-1)}U = \left((\tau-1)\mathcal{V}_Y^{(-1)}\right)\tau U + \mathcal{V}_Y^{(-1)}(\tau-1)U$$

and the congruence

$$(\tau-1)U \equiv XY\frac{dU}{dY} \mod XW(\mathfrak{m}_{\tilde{\mathbf{E}}}).$$

Now, let us carry on computations of Lemma 2.7.:

$$X\mathcal{V}_Y^{-1} = \frac{X}{t}(I_h + p^{M-1}u_1)^t\mathcal{V}^D$$

with $u_1 \in \mathfrak{m}_{[\frac{1}{p-1},p]}$. And since $\mathcal{V}^D$ has coefficients in $\mathcal{G}_{[0,p]} \subset A_{crys}$ where $\varphi_{\mathcal{G}}$ and $\varphi$ coincide, the following holds in $\mathcal{G}_{[p/(p-1),p]}$:

$$\begin{aligned}
\varphi_{\mathcal{G}}\left(X\mathcal{V}_Y^{-1}\right) &= \varphi_{\mathcal{G}}\left(\frac{X}{t}\right)\left(I_h + p^M\varphi_{\mathcal{G}}(v_1)\right)\varphi_{\mathcal{G}}\left({}^t\mathcal{V}^D\right) \\
&= \varphi_{\mathcal{G}}\left(\frac{X}{t}\right)\left(I_h + p^M\varphi_{\mathcal{G}}(v_1)\right) p\ {}^t\mathcal{V}^D\mathcal{E}^{-1}\begin{pmatrix}\frac{1}{p}I_d & 0\\ 0 & I_{h-d}\end{pmatrix} \\
&= \varphi_{\mathcal{G}}\left(\frac{X}{t}\right)(I_h + p^M\varphi_{\mathcal{G}}(v_1))(I_h - p^M v)pt\mathcal{V}_Y^{-1}\mathcal{E}^{-1}\begin{pmatrix}\frac{1}{p}I_d & 0\\ 0 & I_{h-d}\end{pmatrix} \\
&= \varphi_{\mathcal{G}}(X)\left(\mathcal{V}_Y^{-1}\mathcal{E}^{-1} + p^M\tilde{v}\right)\begin{pmatrix}\frac{1}{p}I_d & 0\\ 0 & I_{h-d}\end{pmatrix}
\end{aligned}$$

where $\tilde{v} = \left(\varphi_{\mathcal{G}}(v_1) - v - p^M\varphi_{\mathcal{G}}(v_1)v\right)\mathcal{V}_Y^{-1}\mathcal{E}^{-1}$.
Let us clarify these computations:

$$v, v_1 \in \frac{1}{p}\mathfrak{m}_{[1/(p-1),p]},$$

so that

$$\varphi_{\mathcal{G}}(v_1) \in \frac{1}{p}\mathfrak{m}_{[p/(p-1),p]}.$$

Thus

$$p^M v\varphi_{\mathcal{G}}(v_1) \in \frac{1}{p}\mathfrak{m}_{[p/(p-1),p]}$$

and finally,

$$\varphi_{\mathcal{G}}(v_1) - v - p^M\varphi_{\mathcal{G}}(v_1)v \in \frac{1}{p}\mathfrak{m}_{[p/(p-1),p]}.$$

Hence, since $\mathcal{V}_Y^{-1} \in \frac{1}{Y^{ep/(p-1)}}\mathcal{G}_{[1,p]}$,

$$p\tilde{v} \in \frac{1}{Y^{ep/(p-1)}}\mathfrak{m}_{[p/(p-1),p]}.$$

Finally since

$$\varphi_{\mathcal{G}}(X) \in pX\mathcal{G}_{[0,p]}$$

and

$$X \in Y^{ep/(p-1)}\mathcal{G}_{[p/(p-1),\infty[}$$

we deduce the result. $\Diamond$

Remark that

$$\varphi\left(XY \circ \frac{d}{dY}\right) = \frac{\varphi(X)}{p}Y\frac{d}{dY} \circ \varphi$$

and

$$u\frac{d}{dY}\begin{pmatrix} l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta)\end{pmatrix} \in \mathfrak{m}_{[p/(p-1),p]}$$

so that we compute modulo $p^M\mathfrak{m}_{[p/(p-1),p]}$:

$$\begin{aligned}\varphi_{\mathcal{G}}\left(XY\mathcal{V}_Y^{-1}\frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}\right) &\equiv \frac{\varphi(X)}{p}Y\mathcal{V}_Y^{-1}\frac{d}{dY}\mathcal{E}^{-1}\begin{pmatrix}\frac{1}{p}I_d & 0\\ 0 & I_{h-d}\end{pmatrix}\varphi\begin{pmatrix}l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}\\ &\equiv \frac{\varphi(X)}{p}Y\mathcal{V}_Y^{-1}\frac{d}{dY}\begin{pmatrix}\frac{\mathcal{A}}{p}\circ l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}.\end{aligned}$$

This yields to

$$\begin{aligned}&\varphi\left(XY\mathcal{V}_Y^{(-1)}\frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}\right)\\ =\ &\varphi_{\mathcal{G}}\left(XY\mathcal{V}_Y^{-1}\frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}\right)+\varphi_{\mathcal{G}}\left(XY\left(\mathcal{V}_Y^{(-1)}-\mathcal{V}_Y^{-1}\right)\frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}\right)\\ =\ &\tfrac{\varphi(X)}{p}Y\mathcal{V}_Y^{-1}\tfrac{d}{dY}\begin{pmatrix}\frac{\mathcal{A}}{p}\circ l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}+p^Mu+\varphi_{\mathcal{G}}\left(XY\left(\mathcal{V}_Y^{(-1)}-\mathcal{V}_Y^{-1}\right)\tfrac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}\right)\end{aligned}$$

with $u\in\mathfrak{m}_{[p/(p-1),p]}$. Write $u=u_1+u_2$ with $u_1\in\frac{X^{p-1}}{p}\mathcal{G}_{[0,p]}$, thus $p^Mu_1\in X\mathfrak{m}_{[0,p]}$ and $u_2\in\mathfrak{m}_{[p/(p-1),\infty[}$. In addition, $\mathcal{V}_Y^{(-1)}-\mathcal{V}_Y^{-1}\in\mathcal{G}_{[0,p]}\otimes\mathbb{Q}_p$ hence

$$\varphi_{\mathcal{G}}\left(XY\left(\mathcal{V}_Y^{(-1)}-\mathcal{V}_Y^{-1}\right)\frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}\right)\in X\mathcal{G}_{[0,p]}\otimes\mathbb{Q}_p.$$

Write moreover

$$\begin{aligned}\frac{\varphi(X)}{p}Y\mathcal{V}_Y^{-1}\frac{d}{dY}\begin{pmatrix}\frac{\mathcal{A}}{p}\circ l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix} &= XY\mathcal{V}_Y^{(-1)}\frac{d}{dY}\begin{pmatrix}\frac{\mathcal{A}}{p}\circ l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}\\ &+\left(\frac{\varphi(X)}{p}-X\right)Y\mathcal{V}_Y^{-1}\frac{d}{dY}\begin{pmatrix}\frac{\mathcal{A}}{p}\circ l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}\\ &+XY\left(\mathcal{V}_Y^{-1}-\mathcal{V}_Y^{(-1)}\right)\frac{d}{dY}\begin{pmatrix}\frac{\mathcal{A}}{p}\circ l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}\end{aligned}$$

where

$$XY\left(\mathcal{V}_Y^{-1}-\mathcal{V}_Y^{(-1)}\right)\frac{d}{dY}\begin{pmatrix}\frac{\mathcal{A}}{p}\circ l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}\in\left(X\mathfrak{m}_{[0,p]}\otimes\mathbb{Q}_p\right)^h$$

and

$$\left(\frac{\varphi(X)}{p}-X\right)Y\mathcal{V}_Y^{-1}\frac{d}{dY}\begin{pmatrix}\frac{\mathcal{A}}{p}\circ l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}\in X\left(\mathfrak{m}_{[0,p]}+p^{M-1}\mathcal{G}_{[p/(p-1),\infty[}\right)^h$$

and it can then be written as $M_1+M_2$ with $M_1$ in $X\mathfrak{m}_{[0,p]}$ and $M_2$ in $p^M\tilde{\mathbf{A}}$.

Eventually,

$$\varphi\left(XY\mathcal{V}_Y^{(-1)}\frac{d}{dY}\begin{pmatrix} l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta)\end{pmatrix}\right) - XY\mathcal{V}_Y^{(-1)}\frac{d}{dY}\begin{pmatrix} \frac{\mathcal{A}}{p}\circ l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta)\end{pmatrix} - p^M M_0$$

belongs to $M_h(X\mathcal{G}_{[0,p]}\otimes\mathbb{Q}_p)$ for some $M_0 \in \tilde{\mathbf{A}}$. Then, since $X\mathcal{G}_{[0,p]}\otimes\mathbb{Q}_p\cap\tilde{\mathbf{A}} = X\tilde{\mathbf{A}}^+$, we deduce that

$$\varphi\left(XY\mathcal{V}_Y^{(-1)}\frac{d}{dY}\begin{pmatrix} l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta)\end{pmatrix}\right) = XY\mathcal{V}_Y^{(-1)}\frac{d}{dY}\begin{pmatrix} \frac{\mathcal{A}}{p}\circ l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta)\end{pmatrix} \mod X\tilde{\mathbf{A}}^+ + p^M\tilde{\mathbf{A}}$$

which lets us prove the proposition with the computation modulo $X\tilde{\mathbf{A}}^+ + p^M\tilde{\mathbf{A}}$

$$\begin{aligned}(\varphi-1)\left(XY\mathcal{V}_Y^{(-1)}\frac{d}{dY}\begin{pmatrix} l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta)\end{pmatrix}\right) &\equiv XY\mathcal{V}_Y^{(-1)}\frac{d}{dY}\begin{pmatrix} \left(\frac{\mathcal{A}}{p}-1\right)\circ l_{\mathcal{A}}(\beta) \\ 0\end{pmatrix} \\ &\equiv (\tau-1)x\end{aligned}$$

and the fact that the equation $(\varphi-1)Z = \alpha \in X\tilde{\mathbf{A}}^+ + p^M\tilde{\mathbf{A}}$ admits a solution $Z \in X\tilde{\mathbf{A}}^+ + p^M\tilde{\mathbf{A}}$. □

## 2.5 The explicit formula

### 2.5.1 Statement of the theorem

We come now to the proof of the main theorem, the explicit formula for the Hilbert symbol of a formal group.

**Theorem 2.1.**
*Let $\beta \in F(YW[[Y]])$ and $\alpha \in (W[[Y]][\frac{1}{Y}])^\times$. Denote*

$$\mathscr{L}(\alpha) = \left(1-\frac{\varphi}{p}\right)\log\alpha(Y) = \frac{1}{p}\log\frac{\alpha(Y)^p}{\alpha^\varphi(Y^p)} \in W[[Y]].$$

*Then the Hilbert symbol $(\alpha(\pi),\beta(\pi))_{F,M}$ has coefficients in the basis $(o_M^1,\dots,o_M^h)$:*

$$(\mathrm{Tr}_{W/\mathbb{Z}_p}\circ\mathrm{Res}_Y)\mathcal{V}_Y^{-1}\left(\begin{pmatrix}\left(1-\frac{\mathcal{A}}{p}\right)\circ l_{\mathcal{A}}(\beta) \\ 0\end{pmatrix} d_{\log}\alpha(Y) - \mathscr{L}(\alpha)\frac{d}{dY}\begin{pmatrix}\frac{\mathcal{A}}{p}\circ l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta)\end{pmatrix}\right).$$

### 2.5.2 Proof of Theorem 2.1.

We use the fact that if $\eta \in H^1(K,\mathbb{Z}/p^M\mathbb{Z})$ and $r(x) \in G_K^{ab}$ is the image by the reciprocity isomorphism $x \in K$ then

$$\mathrm{inv}_K(\partial x \cup \eta) = \eta(r(x)).$$

From Proposition 1.3., $\partial\alpha(\pi)$ corresponds to a triple $(x,y,z)$ congruent modulo $XYW[[X,Y]]$ to

$$\left(-\frac{s(Y)}{X}-\frac{s(Y)}{2},\ 0,\ Y d_{\log}S(Y)\right)\otimes\varepsilon.$$

We compute its cup-product with the image

$$(x',0,z')$$

in $H^1(K,F[p^M])$ of $\theta(\beta)$ given by Proposition 2.3. where we recall that

$$x'=\mathcal{V}_Y^{(-1)}\begin{pmatrix}(\frac{\mathcal{A}}{p}-1)\circ l_{\mathcal{A}}(\beta)\\ 0\end{pmatrix}$$

and

$$z'\equiv XY\mathcal{V}_Y^{(-1)}\frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}\quad\bmod XW(\mathfrak{m}_{\tilde{\mathbf{E}}}).$$

We get the triple $(a,b,c)$ where:

$$a=y\mathcal{V}_Y^{-1}\begin{pmatrix}(\frac{\mathcal{A}}{p}-1)\circ l_{\mathcal{A}}(\beta)\\ 0\end{pmatrix}\in W(\mathfrak{m}_{\tilde{\mathbf{E}}})$$

because Proposition 1.3. says that $y\in XYW[[X,Y]]$ and Lemma 2.7. that $XY\mathcal{V}_Y^{(-1)}$ has coefficients in $W(\mathfrak{m}_{\tilde{\mathbf{E}}})+p^M\tilde{\mathbf{A}}$. Moreover,

$$c=-y\otimes\gamma z'+\sum_{n\geq 1}\binom{\chi(\gamma)}{n}\sum_{k=1}^{n-1}C_{n-1}^k(\tau-1)^{k-1}z\otimes\tau^k(\tau-1)^{n-1-k}z'\in W(\mathfrak{m}_{\tilde{\mathbf{E}}})$$

because $y,z,z'\in W(\mathfrak{m}_{\tilde{\mathbf{E}}})$. Finally,

$$b=z\otimes\tau x'-x\otimes\varphi z'$$

and

$$z\otimes\tau x'=(\tau-1)(\log(S(Y))/t+\mu)\ \tau\left(\mathcal{V}_Y^{(-1)}\begin{pmatrix}(\frac{\mathcal{A}}{p}-1)\circ l_{\mathcal{A}}(\beta)\\ 0\end{pmatrix}\right)\otimes\varepsilon.$$

On the one hand

$$(\tau-1)(\log(S(Y))/t+\mu)\equiv Y d_{\log}F(Y)\quad\bmod XYW[[X,Y]]$$

and on the other hand, Lemma 2.8. says $\tau\left(\mathcal{V}_Y^{(-1)}\begin{pmatrix}(\frac{\mathcal{A}}{p}-1)\circ l_{\mathcal{A}}(\beta)\\ 0\end{pmatrix}\right)$ is congruent modulo $XYW[[X,Y]]$ to

$$\mathcal{V}_Y^{(-1)}\begin{pmatrix}(\frac{\mathcal{A}}{p}-1)\circ l_{\mathcal{A}}(\beta)\\ 0\end{pmatrix}+XY\mathcal{V}_Y^{(-1)}\frac{d}{dY}\begin{pmatrix}(\frac{\mathcal{A}}{p}-1)\circ l_{\mathcal{A}}(\beta)\\ 0\end{pmatrix}.$$

Thus, since $XY\mathcal{V}_Y^{(-1)}$ has coefficients in $W(\mathfrak{m}_{\tilde{\mathbf{E}}}) + p^M\tilde{\mathbf{A}}$,

$$z \otimes \tau x' \equiv Y\mathcal{V}_Y^{(-1)} \begin{pmatrix} (\frac{\mathcal{A}}{p} - 1) \circ l_{\mathcal{A}}(\beta) \\ 0 \end{pmatrix} d_{\log} S(Y) \mod W(\mathfrak{m}_{\tilde{\mathbf{E}}}).$$

Finally,

$$-x \otimes \varphi z' = \left( -\frac{s(Y)}{X} - \frac{s(Y)}{2} \right) z' \otimes \varepsilon$$

and since

$$z' \equiv XY\mathcal{V}_Y^{(-1)} \frac{d}{dY} \begin{pmatrix} l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta) \end{pmatrix} \mod XW(\mathfrak{m}_{\tilde{\mathbf{E}}}),$$

we get the congruence

$$-x \otimes \varphi z' \equiv Y s(Y) \mathcal{V}_Y^{(-1)} \frac{d}{dY} \begin{pmatrix} l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta) \end{pmatrix} \mod W(\mathfrak{m}_{\tilde{\mathbf{E}}}).$$

The triple $(a, b, c)$ is eventually congruent modulo $W(\mathfrak{m}_{\tilde{\mathbf{E}}})$ to

$$\left( 0, Y\mathcal{V}_Y^{(-1)} \left( \begin{pmatrix} (\frac{\mathcal{A}}{p} - 1) \circ l_{\mathcal{A}}(\beta) \\ 0 \end{pmatrix} d_{\log} S(Y) + \frac{d}{dY} \begin{pmatrix} l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta) \end{pmatrix} \frac{1}{p} \log \frac{S(Y)^p}{S(Y^p)} \right) \otimes \varepsilon, 0 \right).$$

The theorem follows then from the lemma:

**Lemma 2.9.**
*Let $C = C_{\varphi,\gamma,\tau}(\tilde{\mathbf{A}}_L(1))$ be the complex computing Galois cohomology of $\mathbb{Z}_p(1)$.*

1. *Let $f(Y) = \sum_{n>0} \frac{a_n}{Y^n} \in M_h(\tilde{\mathbf{A}})$ be the principal part of a series $\mathcal{V}_Y^{(-1)} g(Y)$ with $g(Y)$ having coefficients in $W[[Y]]$. Then there exists a triple $(x_1, x_2, 0)$ with coefficients in $\in W(\mathfrak{m}_{\tilde{\mathbf{E}}})$ such that*

   $$(x_1, x_2 + f(Y) \otimes \varepsilon, 0) \in B^2(C).$$

   *In other words its image in $H^2(K, \mathbb{Z}_p(1))$ is zero.*

2. *Let $(x, y, z) \in Z^2(C)$ with $x, y, z \in W(\mathfrak{m}_{\tilde{\mathbf{E}}})(1)$ then*

   $$(x, y, z) \in B^2(C).$$

3. *Let $w \in W$ then*

   $$(0, w \otimes \varepsilon, 0) \in Z^2(C)$$

   *and its image through the reciprocity isomorphism is $\mathrm{Tr}_{W/\mathbb{Z}_p}(w)$.*

*Proof of the Lemma:* Put

$$w_n = \frac{1}{Y^n\left((1+X)^{-n}-1\right)} + \frac{1}{2Y^n} \in \tilde{\mathbf{A}}_L.$$

Then

$$(\tau-1)w_n = \frac{1}{Y^n} + \frac{1}{2}(\tau-1)\frac{1}{Y^n} \tag{2.13}$$

and

$$\begin{aligned}
\gamma\left(\frac{\varepsilon}{Y^n\left((1+X)^{-n}-1\right)}\right) &= \frac{\chi(\gamma)\varepsilon}{Y^n\left((1+X)^{-\chi(\gamma)n}-1\right)} \\
&= \chi(\gamma)\delta^{-1}\left(\frac{\varepsilon}{Y^n\left((1+X)^{-n}-1\right)}\right).
\end{aligned}$$

The Taylor expansion

$$\delta^{-1} = \chi(\gamma) - \frac{\chi(\gamma)(\chi(\gamma)-1)}{2}(\tau-1) + (\tau-1)^2 g(\tau-1)$$

where $g(\tau-1)$ is a power series in $\tau-1$ yields to the relation

$$(\gamma-1)w_n \otimes \varepsilon = g(\tau-1)(\tau-1)\frac{1}{Y^n}. \tag{2.14}$$

From Lemma 2.8., we know $(\tau-1)\mathcal{V}^{(-1)}U$ for $U \in W[[Y]]$ has coefficients in $W(\mathfrak{m}_{\tilde{\mathbf{E}}})$. Relation (2.13) then shows that

$$(\tau-1)\sum_{n>0} a_n w_n = f(Y) \mod W(\mathfrak{m}_{\tilde{\mathbf{E}}})$$

and Relation (2.14) that

$$(\gamma-1)\sum_{n>0} a_n w_n = 0 \mod W(\mathfrak{m}_{\tilde{\mathbf{E}}})$$

which proves that the coboundary image of triple $(\sum_{n>0} a_n w_n, 0, 0)$ in $H^2(C)$ has the desired form, hence 1.
To show 2. we have to solve for $x, y, z \in W(\mathfrak{m}_{\tilde{\mathbf{E}}})(1)$ the system

$$\begin{aligned}
x &= (\gamma-1)u + (1-\varphi)v \\
y &= (\tau-1)u + (1-\varphi)w \\
z &= (\tau^{\chi(\gamma)}-1)v + (\delta-\gamma)w
\end{aligned}$$

Consider therefore $v,\ w \in W(\mathfrak{m}_{\tilde{\mathbf{E}}})(1)$ solutions of

$$\begin{aligned}
x &= (\varphi-1)v \\
y &= (\varphi-1)w
\end{aligned}$$

which exist, and are unique since $\varphi-1$ is bijective on $W(\mathfrak{m}_{\tilde{\mathbf{E}}})(1)$. Then, by combining these equations with the ones of the system, we get

$$(\varphi-1)((\tau^{\chi(\gamma)}-1)v+(\delta-\gamma)w)=-(\tau^{\chi(\gamma)}-1)x-(\delta-\gamma)y=(\varphi-1)z.$$

But since $z$ and $(\tau^{\chi(\gamma)}-1)v+(\delta-\gamma)w$ are elements of $W(\mathfrak{m}_{\tilde{\mathbf{E}}})(1)$ where $(\varphi-1)$ is injective, the equality

$$z=(\tau^{\chi(\gamma)}-1)v+(\delta-\gamma)w$$

holds ; $(x,y,z)$ is then a coboundary, the image of $(0,v,w)$.
Finally, for Point 3., remark that

$$(0,w\otimes\varepsilon,0)=(0,0,1\otimes\varepsilon)\cup(w,0,0).$$

Proposition 1.3. says $(0,0,1\otimes\varepsilon)$ is the image through the Kummer map of $\pi$ a uniformizer of $K$. (To see this, take $F(Y)=Y$.) In addition $(0,w\otimes\varepsilon,0)$ corresponds from Theorem 1.3. to the character $\eta$ of $G_K$ defined in the following way: choose $b\in\tilde{\mathbf{A}}$ such that $(\varphi-1)b=w$, then

$$\forall g\in G,\ \eta(g)=(1-g)b.$$

Remark that since $w\in W$, we can choose $b\in W^{nr}$ and that the image through the Kummer map of a uniformizer is the Frobenius $\mathrm{Frob}_K$, thus the image through reciprocity isomorphism of $(0,w\otimes\varepsilon,0)$ is

$$(1-\mathrm{Frob}_K)b=(1-\varphi^{f_K})b=(1+\varphi+\cdots+\varphi^{f_K-1})w=\mathrm{Tr}_{W/\mathbb{Z}_p}w$$

where $f_K=f(K/\mathbb{Q}_p)$, which proves the lemma. $\Diamond$

We prove then the theorem by remarking, from the congruence shown above, that the triple $(a,b,c)$ can be written as a sum of a triple $(0,g(Y),0)$ where $g$ is the negative part of a vector series in $Y$ and then is zero in $H^2(K,\mathbb{Z}/p^M\mathbb{Z})$, of a triple with coefficients in $W(\mathfrak{m}_{\tilde{\mathbf{E}}})(1)$, then also a coboundary because of the lemma above and finally a triple $(0,w\otimes\varepsilon,0)$ where $w$ is the constant term of the vector series

$$Y\mathcal{V}_Y^{(-1)}\left(\begin{pmatrix}(\frac{\mathcal{A}}{p}-1)\circ l_{\mathcal{A}}(\beta)\\0\end{pmatrix}d_{\log}\alpha(Y)+\frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\m_{\mathcal{A}}(\beta)\end{pmatrix}\frac{1}{p}\log\frac{\alpha(Y)^p}{\alpha(Y^p)}\right)$$

hence the residue of

$$\mathcal{V}_Y^{-1}\left(\begin{pmatrix}\left(\frac{\mathcal{A}}{p}-1\right)\circ l_{\mathcal{A}}(\beta)\\0\end{pmatrix}d_{\log}\alpha(Y)+\frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\m_{\mathcal{A}}(\beta)\end{pmatrix}\frac{1}{p}\log\frac{\alpha(Y)^p}{\alpha(Y^p)}\right).$$

The only term with a non zero contribution is then the residue, and that contribution is, according to the lemma, given by the trace, which completes the proof. $\square$

## References


[Abr95] V. A. Abrashkin, *A ramification filtration of the Galois group of a local field. II*, Trudy Mat. Inst. Steklov. **208** (1995), no. Teor. Chisel, Algebra i Algebr. Geom., 18–69, Dedicated to Academician Igor′ Rostislavovich Shafarevich on the occasion of his seventieth birthday (Russian). MR MR1730256 (2000m:11117a)

[Abr97] ______, *Explicit formulas for the Hilbert symbol of a formal group over Witt vectors*, Izv. Ross. Akad. Nauk Ser. Mat. **61** (1997), no. 3, 3–56. MR MR1478558 (98j:14060)

[Ben97] Denis Benois, *Périodes p-adiques et lois de réciprocité explicites*, J. Reine Angew. Math. **493** (1997), 115–151. MR MR1491810 (98m:11128)

[Ben00] ______, *On Iwasawa theory of crystalline representations*, Duke Math. J. **104** (2000), no. 2, 211–267. MR MR1773559 (2001m:11208)

[Ber02] Laurent Berger, *Représentations p-adiques et équations différentielles*, Invent. Math. **148** (2002), no. 2, 219–284. MR MR1906150 (2004a:14022)

[Ber03] ______, *Bloch and Kato's exponential map: three explicit formulas*, Doc. Math. (2003), no. Extra Vol., 99–129 (electronic), Kazuya Kato's fiftieth birthday. MR MR2046596 (2005f:11268)

[Ber04] ______, *Limites de représentations cristallines*, Compos. Math. **140** (2004), no. 6, 1473–1498. MR MR2098398 (2006c:11138)

[Bre99] Christophe Breuil, *Une application de corps des normes*, Compositio Math. **117** (1999), no. 2, 189–203. MR MR1695849 (2000f:11157)

[CC98] Frédéric Cherbonnier and Pierre Colmez, *Représentations p-adiques surconvergentes*, Invent. Math. **133** (1998), no. 3, 581–611. MR MR1645070 (2000d:11146)

[CC99] ______, *Théorie d'Iwasawa des représentations p-adiques d'un corps local*, J. Amer. Math. Soc. **12** (1999), no. 1, 241–268. MR MR1626273 (99g:11141)

[Col81] Robert F. Coleman, *The dilogarithm and the norm residue symbol*, Bull. Soc. Math. France **109** (1981), no. 4, 373–402. MR MR660143 (84j:12017)

[Col92] Pierre Colmez, *Périodes p-adiques des variétés abéliennes*, Math. Ann. **292** (1992), no. 4, 629–644. MR MR1157318 (93d:14062)

[FL82] Jean-Marc Fontaine and Guy Laffaille, *Construction de représentations p-adiques*, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 4, 547–608 (1983). MR MR707328 (85c:14028)

[Fon77] Jean-Marc Fontaine, *Groupes p-divisibles sur les corps locaux*, Société Mathématique de France, Paris, 1977, Astérisque, No. 47-48. MR MR0498610 (58 #16699)

[Fon90] ______, *Représentations p-adiques des corps locaux. I*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 249–309. MR MR1106901 (92i:11125)

[Fon94] ______, *Le corps des périodes p-adiques*, Astérisque (1994), no. 223, 59–111, With an appendix by Pierre Colmez, Périodes $p$-adiques (Bures-sur-Yvette, 1988). MR MR1293971 (95k:11086)

[Her98] Laurent Herr, *Sur la cohomologie galoisienne des corps p-adiques*, Bull. Soc. Math. France **126** (1998), no. 4, 563–600. MR MR1693457 (2000m:11118)

[Her01] ______, *Une approche nouvelle de la dualité locale de Tate*, Math. Ann. **320** (2001), no. 2, 307–337. MR MR1839766 (2002d:11136)

[Hon70] Taira Honda, *On the theory of commutative formal groups*, J. Math. Soc. Japan **22** (1970), 213–246. MR MR0255551 (41 #212)

[Kis06] Mark Kisin, *Crystalline representations and $F$-crystals*, Algebraic geometry and number theory, Progr. Math., vol. 253, Birkhäuser Boston, Boston, MA, 2006, pp. 459–496. MR MR2263197 (2007j:11163)

[Sch06] Anthony J. Scholl, *Higher fields of norms and $(\phi, \Gamma)$-modules*, Doc. Math. (2006), no. Extra Vol., 685–709 (electronic). MR MR2290602 (2007k:11199)

[Sen80] Shankar Sen, *On explicit reciprocity laws*, J. Reine Angew. Math. **313** (1980), 1–26. MR MR552459 (81a:12018)

[Ser68] Jean-Pierre Serre, *Corps locaux*, Hermann, Paris, 1968, Deuxième édition, Publications de l'Université de Nancago, No. VIII. MR MR0354618 (50 #7096)

[Ser94] ______, *Cohomologie galoisienne*, fifth ed., Lecture Notes in Mathematics, vol. 5, Springer-Verlag, Berlin, 1994. MR MR1324577 (96b:12010)

[TR08] Floric Tavares Ribeiro. $(\varphi, \Gamma)$-modules et loi explicite de réciprocité. *Thèse de Doctorat de l'Université de Franche-Comté.*

[VD00] S. V. Vostokov and O. V. Demchenko, *An explicit formula for the Hilbert pairing of formal Honda groups*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **272** (2000), no. Vopr. Teor. Predst. Algebr i Grupp. 7, 86–128, 346. MR MR1811794 (2002k:11213)

[Ven03] Otmar Venjakob, *A non-commutative Weierstrass preparation theorem and applications to Iwasawa theory*, J. Reine Angew. Math. **559** (2003), 153–191, With an appendix by Denis Vogel. MR MR1989649 (2004e:11123)

[Vos00] Sergei V. Vostokov, *Explicit formulas for the Hilbert symbol*, Invitation to higher local fields (Münster, 1999), Geom. Topol. Monogr., vol. 3, Geom. Topol. Publ., Coventry, 2000, pp. 81–89 (electronic). MR MR1804922 (2002a:11138)

[Wac96] Nathalie Wach, *Représentations $p$-adiques potentiellement cristallines*, Bull. Soc. Math. France **124** (1996), no. 3, 375–400. MR MR1415732 (98b:11119)

[Win83] Jean-Pierre Wintenberger, *Le corps des normes de certaines extensions infinies de corps locaux; applications*, Ann. Sci. École Norm. Sup. (4) **16** (1983), no. 1, 59–89. MR MR719763 (85e:11098)

Floric Tavares Ribeiro
Département de Mathématiques, Université de Franche-Comté
16 route de Gray, 25000 Besançon, France
*email:* ftavares@univ-fcomte.fr